THE APPLICATION OF EUCLIDEAN SPACE FOR THE ASSESSMENT OF RANDOMLY DISTRIBUTED RADIOACTIVE MATERIAL UTILIZING MULTIPLE DETECTOR MEASUREMENTS

By

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Although the pioneer of this field, Dr. Yakov Ben-Haim, is currently focused on research areas different than the subject of this dissertation, he has never failed to answer questions, clear misconceptions, review half-formed ideas and encourage the author's attempts toward contributing to Dr Ben-Haim's previously published work.

Finally, because the above thanks to his chair seem embarrassing insufficient, the author again thanks Dr. Alan M. Jacobs for introducing the subject and for his invaluable guidance on dealing with the uncertainties present in both unknown spatial distributions and the pursuit of a Ph.D.

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KEY TO SYMBOLS

SYMBOL	MEANING
$a=(a_1,a_2,\ldots a_N)$	Vector used for defining the slopes of an N-1 dimension hyperplane in N space
$<a,f>$	Scalar product of a and f
b	Scalar
C	An arbitrary set or a complete response set
d_f	Distance from the origin to a point f
d_{fg}	Distance between the points f and g
$f=(f_1,f_2,\ldots,f_N)$	A vector or point in N space with components $f_1,f_2,\ldots,f_N$
F	An arbitrary set of a point source response set
m	Amount of radioactive material present
N	The dimension of the Euclidian space or the number of detectors present
$r(x,y,z)$	Density of activity at point (x,y,z)
(r,θ,z)	Cylindrical coordinates in sample space
RMR	Relative mass resolution
$s=(s_1,s_2,\ldots,s_N)$	Slopes of a line in N space
S	An arbitrary set

(x,y,z)	Cartesian coordinates of a point in sample space
X,Y,Z	The spatial domain of the radioactive material
$r(x,y,z)$	Density of activity at point (x,y,z)

Abstract of Dissertation Presented to the Graduate School
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Major Department: Nuclear Engineering Sciences

If a detector response is proportional to a fixed distribution of radioactive material, the measured radioactivity can be assessed from a single detector response. Unknown or random spatial distributions of activity, however, results in an uncertainty associated with the assessment. This spatial assessment uncertainty can be reduced by utilizing multiple detector measurements.

Conventional techniques such as averaging or summing of multiple detector responses combined possibly with rotation and segmentation of the source container avoid the complexity of directly interpreting a multiple detector measurement and result in a loss of information by ignoring the additional dimensions of the problem. The vector representation of a multiple detector response allows for the maximum utilization

of a limited amount of information. By considering the mathematical relationships between the detectors, formulations have been developed in this dissertation which allow for the assessment of both the amount and associated spatial uncertainty of the radioactive material measured from N detector responses.

The analysis of the proportional relationships between the detector responses results in the following: the calculation of the possible activities from a detector response of unknown activity and distribution, the calculation of the largest possible uncertainty from N measurements for a constrained distribution of activity, the property that local optimization over an unconstrained distribution is a global optimization. The analysis of the first and second derivatives of a response sets results in a technique for reducing both the number of optimization steps and the number of independent variables necessary for the calculation of the largest uncertainty.

Increasing the number of symmetric detectors surrounding a container results in the largest uncertainty approaching some asymptotic limit. A relationship between an infinite number of symmetric measurements and a definite integral is developed allowing the solution to the asymptotic limit to be obtained by analytical means.

The optimization of a detector system by minimizing the largest uncertainty with respect to detector position is

dependent on the amount of activity actually present. Utilizing the above mathematical relationships, a detector system which optimizes the $(N+1)$th detector based on the results of the interpretation of the previous N detectors is developed.

CHAPTER 1
INTRODUCTION

The focus of this dissertation is the application of dimensional space for the assay of randomly distributed radioactive material. The presence of radioactive material can be determined by performing measurements with suitable detectors (Knoll 1979). The utilization of these detector responses for the assessment of a measured amount of radioactive material is well understood for known source distributions of the radioactive material. However, uncertainty concerning the radioactive material distribution, possibly due to some inherent random property or lack of sufficient information on the part of the assessor, poses additional complications.

Given the assumptions that the spatial distribution of the radioactive material within the bounds of a container is unknown and that a limited number of radiation detectors are present, it is difficult and often impossible to construct the actual spatial configuration of the source present. In many cases, different amounts of radioactive material with different spatial distributions can result in identical detector responses. The methodology presented in this

dissertation is applicable to the problem of effectively utilizing the limited information container from a relatively small number of detector measurements and not on the more sophisticated problem of mapping the concentrations of the radioactivity present.

The introduction of mathematical methods for the assay of randomly distributed radioactive material has been limited. The difficulties of assessing both the amount and the inherent uncertainties of the radioactive material in the presence of random or unknown distributions has been suggested as an area requiring prior information and expert judgment (Sampson 1992). If the distribution of the radioactive material can be assumed, probabilistic methods can be applied (Ben-Haim 1983). Techniques such as rotation of the source container, segmentation of the measurement volume seen by the detector and linear combinations of detector responses have been introduced as methods for reducing the assessment uncertainty for such configurations as a point source and a small cylinder source within a waste container (Martens and Fliss 1990).

The application of dimensional space has previously been introduced as a tool for a calculating a performance criterion called the relative mass resolution (RMR). The RMR is defined as the ratio of the maximum amount of radioactivity compared to a fixed amount of activity that could result in an identical set of detector responses. This application, however, does not extend to the problems of either

interpreting multiple detector responses in the presence of randomly distributed radioactive material (Ben-Haim 1983) or assessing the actual uncertainty from a given set of detector measurements. In addition, the search method suggested for the RMR possesses a number of limitations. The search method is a local optimization method (Scott 1991) and is valid only for cases where the detector response set is convex (Ben-Haim 1985a, 1985b). Appendix A contains a number of definitions concerning *N*-dimensional space and convex set.

The above limitations are addressed within this dissertation by extending the calculational methods. A solution to the problem of interpreting multiple detector measurements in the presence of spatial uncertainty is proposed. A global search technique for convex sets is presented. A complementary derivative technique, which reduces the number of search variables, is added to the existing method for calculating the RMR (Ben-Haim 1983). A method of analyzing the asymptotic limit of an increasing number of detectors is presented.

Incorporated within the development of the RMR is the concept of representing the set of detector responses from unconstrained source distributions by the set of detector responses from point source distributions. A number of examples and proofs expanding on the subject of convex sets exist (Ben-Haim 1985b). A related dissertation (Chan 1988) was

mostly an application which relied heavily on previously formulated concepts and equations.

The research pursued and described in this dissertation, while unified by the concept of dimensional space, has taken a number of directions. Published literature in the field (Ben-Haim 1985b) mentions a number questions and problems that exist. Five problems (Ben-Haim 1985b, p. 298) for future research were briefly discussed. Solutions to two of the problems (nonconvex response sets and asymptotic designs) are given within the dissertation. Contributions to a third problem (adaptive assay-advanced concepts) are given in the Appendix B. Along with the unresolved issues that can be found in the published literature, the practioner of the assay of spatially random material is presented with a number of immediate concerns. The first concern is that if a detector system is optimized with the aid of the RMR, the methodology presently available for interpretation of the detector measurements requires the knowledge of prior probability distributions (Ben-Haim 1985b). This probabilistic interpretation is more complex than the analysis necessary to optimize the detector system. In addition, while the RMR has been demonstrated as a tool for comparing detector systems to each other, it is not apparent that a detector system optimized with the aid of the RMR is optimal if another methodology is utilized for interpretation. Since the practitioner also confronts an increasing number of search

variables as the number of measurements increases, investigations have been performed in the area of variable reduction and optimization of search techniques.

The structure of the dissertation is divided into methodologies. Chapter 3 extends the application of dimensional space to the problem of interpreting detector responses. The resulting method, called the ray method, can be utilized to assess the amounts of radioactive material that could result in a given N detector response. Chapter 4 extends the ability to calculate the RMR. The use of supporting planes was previously developed (Ben-Haim 1983) as a method for the calculation of the RMR of a convex response set utilizing only the fundamental set that generated the convex set. The contributions to this method involve the ability to efficiently calculate the RMR. For some problems the weight coefficients can be substituted for derivatives, thus reducing the number of variables of the search space. In addition, the difficulties of utilizing the existing search method for calculating the RMR is discussed and alternative strategies are developed. The ray method is applied to the solution of the RMR in the presence of nonconvex sets and to the development of a global optimization method for the solution of the RMR in the presence on convex sets. Chapter 5 develops a method for determining the asymptotic limit of the RMR as the number of symmetric detectors approach infinity in the presence of a fixed number of nonsymmetric detectors.

The following is a summary of the fundamental terms and theory that have previously been developed (Ben-Haim 1985a). Each detector response represents one axis on a Cartesian coordinate system. For example, Figure 1-1 illustrates a system that has three detectors with responses f_1, f_2 and f_3. A coordinate system is constructed with the x-axis representing f_1, the y-axis f_2 and the z-axis f_3. A response point is the N detector response to a specified distribution of source activity. Figure 1-2 represents the above detector system responding to a given amount of radioactivity with a given source distribution.

Assuming a constant attenuation coefficient, if the location of a point source of radioactivity continuously changes, the representative point in the response space also changes as demonstrated in Figure 1-3. The set of all possible locations of a single point of radioactivity of unit strength over the source container is called the point source response set. The complete response set is defined as the set of all detector responses from any spatial distribution with one unit of source material.

The application of much of the methodology presented is based on the assumption that the detector responses are proportional to the amount of activity present and that the self-attenuation of the source is negligible. To allow for analytical solutions in the examples presented, the calculations are simplified by a number of additional

assumptions. The detectors respond as point detectors. Attenuation of both the source container and the medium between the container and the detector is considered negligible. The response from the nth detector to a point source is given as $f_n=1/d_n^2$ where d_n is the distance from the point source to the detector.

While the main purpose of the dissertation is to extend the application of dimensional space to the problems of radioactive material assessment, a secondary purpose is to increase the understanding of the subject for the practitioner. The utilization of dimensional space for the assay of radioactive material is not common. In addition, the emphasis of this dissertation is on the understanding of the mathematical tools possible from applying dimensional space rather than on the utilization of supporting planes or convex sets. The result of such an approach is that while a number of innovative ideas are presented from a unified perspective, significant work concerning concepts previously developed, such as the RMR and convex sets (Ben-Haim 1984, 1985a) has been summarized.

A number of examples are presented which illustrate the ideas and the problem solving techniques formulated. For graphical reasons two detectors are usually stipulated. Several detector, source container configurations are repeated throughout the examples. The one dimensional source container presented is a line segment on the x-axis from $x=-0.5$ to $x=0.5$

with detectors located on the x-axis at $x=-1$ and $x=1$. The two dimensional source container is a circle of radius $r=0.5$ centered on the origin. The three dimensional source container stipulated is a cube of length one centered on the origin with sides perpendicular to the x,y and z axes.

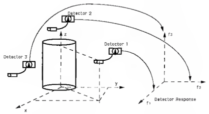

Figure 1-1. Three detector set-up and the resulting detector response coordinate system.

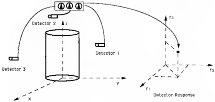

Figure 1-2. Three detector set-up and a resulting response point in detector response space.

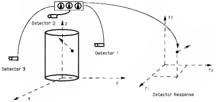

Figure 1-3. A change in the location of the source and the resulting change in the point response.

CHAPTER 2
FUNDAMENTAL CONCEPTS

Application of Vector Space for Detector Responses

Assuming that the detector responses are proportional to the amount of radioactive material present and that the radioactive material distribution is known, the use of additional detectors at different spatial locations does not provide additional information necessary to assess the amount of activity present. If the statistical uncertainty from the detector responses is ignored, a single detector response and the spatial configuration of the radioactivity is sufficient to calculate the amount of radioactivity present. A common example is the use of a single detector response at a specific location to determine the activity of a waste container. The contents of the waste container are assumed to be homogeneous, thus allowing a calculated value without uncertainty to represent the activity present.

Uncertainty concerning the source configuration results in uncertainty concerning the interpretation of a detector response. When attempting to assess an amount of radioactivity present as illustrated in Figure 2-1, one unit of activity at a specified location could result in the same response as two units of activity at a different location. Figure 2-2

demonstrates the ability of a two detector system to distinguish between these two distinct cases of source activity and location.

The detector system in Figure 2-2 can be described as a two variable array (f_1, f_2), where f_1 is the response of the first detector and f_2 is the response of the second detector. The utilization of the two detector system in Figure 2-2 results in a less uncertain assessment of activity than the one detector system illustrated in Figure 2-1.

If the spatial distribution of the activity is known, the methodology for assessing the radioactive material for a one detector system with a known distribution can be performed by utilizing a previously determined response per activity factor for a particular nuclide. The activity is calculated by dividing the factor into the given detector response.

For a one detector system, if the spatial distribution of the activity is unknown, then the maximum and minimum possible detector responses for one unit of activity are calculated. For the detector system in Figure 2-1, the cases of greatest and least detector response are point sources located at the distances closest to and farthest from the detector. The amount of activity possible from a given detector response is calculated by dividing the given detector response by the minimum and maximum detector responses to obtain the respective maximum and minimum amounts of activity. All other possible amounts of activity that could result from the

extreme detector responses are between these calculated amounts of activity.

When attempting to apply the above one detector intuitive type procedure to a multiple detector system additional questions arise. What is an extreme detector response? How is a response "between" to other extreme responses defined? How is an assessment of activity performed from a multiple detector response? The above questions indicate the failure of attempting to use a strictly intuitive method for multiple detector responses. A more mathematical approach however, is possible and has been previously presented (Ben-Haim 1983). From this approach, which applies N dimensional space to an N detector response system, the concepts of extreme responses and responses between responses are easily extended from a one detector system to a multiple detector system.

The foundation of the mathematical approach is that responses from N different detectors are considered as coordinate axes in N dimensional space. The application of N dimensional space allows for the representation of detector responses as points. A set of points resulting from a specific amount of activity is called a response set. An extreme point is the point of farthest distance from the other points. The property of proportionality of activity allows for algebraic operations to be performed on this set. These operations allow for the interpretation of an N dimensional detector response.

Properties of Response Sets

Complete and Point Response Sets

Before interpreting a given detector response, the set of responses from a specific amount of activity is required to be constructed. If the spatial configuration of the activity is assumed to have a specific distribution, such as a homogeneous distribution, then ignoring statistical uncertainty, the response set will be composed of a point. If, however, the distribution is not known, then the detector response results in a number of points where each point on the response set represents one or more spatial distributions of activity.

Although the location of the source within the container is unknown, the source may be restricted to a particular distribution. For example, a radioactive sample vial or source planchet, while possessing an unknown location within a room or waste container, will still remain in its original shape. Therefore, depending on the restrictions of the source itself, different response sets are possible. The two response sets considered in this dissertation are the point source response set and complete response set. The point source response set results from all responses from a point of unit amount of radioactive material. The complete response set results from all possible distributions of a unit amount of radioactive material. The point source response set can be analytically constructed by forming a parametric set of detector response

equations that are dependent on the spatial coordinates of the point source. For N detectors there are N equations of the form $f_n=f_n(x,y,z)$. The complete response set, however, is more difficult to construct using functional relationships since such a construction would require a systematic search of the different spatial positions possible. Fortunately as summarized in the following section, the complete response set can be derived from the point source response set. In addition, the relation of the complete and point source response sets allows for the complete response set to be considered "between" the point source response set.

<u>Construction of the Complete Response Set</u>

The point source response set F is the set of all detector responses that are possible from one unit of activity of a single point source and is defined as

$$F=\{f:f=f(x,y,z) \text{ for all } x\in X,\ y\in Y \text{ and } z\in Z\} \qquad (2\text{-}1)$$

where X,Y and Z represent the set of all possible locations of a single point source. If this unit point source is divided into J amounts $a_1,a_2,\ldots,a_J$ at respective locations (x_j,y_j,z_j) the detector response to this combination of subunit sources is defined as

$$c=\sum_J a_j f(x_j,y_j,z_j)\text{, where } \sum_J a_j=1. \qquad (2\text{-}2)$$

Due to the formulations that result from the above equation, a careful examination of Equation 2-2 is in order. Equation 2-2, which is the detector response resulting from a group of subunit points sources adding to one, is equivalent to a convex combination of the points of J points

$$f(x_1,y_1,z_1),f(x_2,y_2,z_2),\ldots,f(x_J,y_J,z_J) \qquad (2\text{-}3)$$

This equivalence is the result of the assumption that the detector response is proportional to the amount of activity measured. From Appendix A, the set of all convex combinations of the elements of the point source response set is equivalent to the convex hull of the point source response set. Although the examples throughout the dissertation present a number of simplified formulations for f, the complexity of f and its dependence on variables such as location, attenuation coefficient of the matrix and scattered radiation does not affect the above relationship as long as the response remains proportional to activity.

The boundaries of the complete response set are composed of the point source response set and linear combinations of the point response set. For a two dimensional response space, the boundaries of the complete response set will be the point response set and line segments whose endpoints are elements of the point response set. Example 2-1 formulates a case where the extreme points of the point source response set define the line segment boundary of the complete response set. These endpoints represent the unit source closest to one of the

respective detectors. In general, an N dimensional complete response set will have the boundaries composed of the point response set and sections of N-1 dimensional hyperplanes defined by elements from the point response set. The question as to when is a response "between" other responses has been resolved.

If the number of quantities and respective locations of the subunit point sources, J, approach infinity, then the set of detector responses is represented as

$$C_r=\{c_r : c_r(x,y,z)=\iiint_{XYZ} r(x,y,z)\,f(x,y,z)\,dx\,dy\,dz \text{ for all } r\in R\}$$

where R is the set of all functions integrable on the space (X,Y,Z) for one unit of activity. The above equation is both a infinite extension to the definition of a convex combination and the point kernel representation to a unit amount of radioactivity.

The advantage of demonstrating that the complete response set is the convex hull of the point source response set is that the complete response set can be formulated in terms of the point source response set. The relationship between the sets will prove advantageous when searching over the complete response set. The construction of a single point from the complete response set requires N points from the point source response set. Each of the N points from the point response set has a coefficient. Since the coefficients sum to one, only N-1 coefficients are independent. Therefore 2N-1 independent

variables are required to determine a point in the complete response set. The entire complete response set can be determined by allowing the 2N-1 terms to vary over their respective ranges.

Example 2-1. The radioactivity is confined to the line segment $-0.5 \le x \le 0.5$. The detectors are located on the x axis at $x=-1$ and $x=1$ with response functions

$$f_1 = \frac{1}{(1+x)^2} \text{ and } f_2 = \frac{1}{(1-x)^2} \qquad (2\text{-}5)$$

The complete response is represented as

$$C = \{c : c(x,x',a) = \left(\frac{a}{(1+x)^2} + \frac{1-a}{(1+x')^2}, \frac{a}{(1-x)^2} + \frac{1-a}{(1-x')^2}\right)$$

where $\Sigma\alpha=1$. If $x=0.5$ and $x'=-0.5$ in the above equation, the group of points belonging the complete response set is

$$c(0.5,-0.5,a) = (4-3.556a, 0.4444+3.556a). \qquad (2\text{-}7)$$

The above equation represents the parametric equation of a line segment between the points (4, 0.444) and (0.444, 4). In terms of f_1 and f_2 the equation is written as

$$f_2 = 4.444 - f_1 \text{ where } 0.4444 \le f_1 \le 4 \qquad (2\text{-}8)$$

Since the detector system is two dimensional, the point source response set can be represented graphically. Figure 2-3 illustrates the point source response set. The complete response set can be constructed graphically by creating a line segment at the extremes of the point response set. The

equation of this line is represented in equation 2-7. The area contained by the point source response set and the line segment is the complete response set. Figure 2-4 illustrates the complete response set.

The above example introduces the concept of linear equations, at least partially, bounding a convex set. The hyperplane that bounds a convex set C which is the convex hull of a set F, is called a supporting plane. A convex set will, at each extreme point, have a supporting plane. Since the intersection of convex sets is a convex set, the intersection of the supporting plane H with the convex set C or $C \cap H$, at the point x is itself a convex set. If the point x is a member of the set C, but not the set F, then x is some convex combination of points from F. These points from F define a hyperplane P that either does or does not contain the set $C \cap H$. If P does contain the set $C \cap H$, then the supporting plane does intersect the set F. If P does not contain the set $C \cap H$, then some direction along P will form a parallel hyperplane defined by a point from F. Therefore, the hyperplane H cuts C and does not bound C and H and is not a supporting plane. The supporting plane of C will contain a point from the set F and supporting planes of a convex hull of a set F will intersect the set F. The importance of the above result is that a linear optimization over the complete response set can be performed over the point source response set reducing the number of variable required for a search.

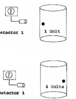

Figure 2-1. One detector system unable to distinguish between different amounts of activity and different source locations.

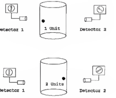

Figure 2-2. Two detector system distinguishing between different amounts of activity and different source locations.

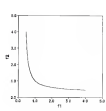

Figure 2-3. Point source response set for a line source container.

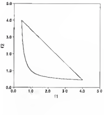

Figure 2-4. Complete response set for a line source container.

CHAPTER 3
THE LINE INTERSECTION (OR RAY) METHOD

Introduction

Previous published research on the assay of randomly distributed radioactive material attempted to fulfill a lacking in the subject which has been summarized as follows: "the theoretical ramifications of spatial uncertainty have never been analyzed in a systematic way nor have been given a unified treatment. Mathematical techniques for rigorously assessing the spatial uncertainty of a proposed assay system design have, until recently, been lacking" (Ben-Haim 1985b, pg.2).

The RMR was developed as a tool for assessing the spatial uncertainty of an detector assay system. As acknowledged, however, "careful design is not enough. The analyst must properly interpret the results of his measurements" (Ben-Haim 1985b, pg.5). The problems of assay system design and interpretation of measurements, however, were given different mathematical treatments (Ben Haim 1985b, pg. 12). While the tools of assay system design were given a deterministic treatment, the interpretation of detector responses was performed utilizing probabilistic methods.

The reason for the separation is, in part, historical. Articles in the early 1980s (Ben-Haim and Elias 1982, Ben-Haim 1983, Ben Haim et al. 1981) explored the use of probabilistic methods to interpret detector readings. The probabilistic interpretation method was incorporated with the idea of constructing sets and developing the concept of the RMR (Ben-Haim 1983). A number of articles followed that incorporated the idea of the RMR in the existing set theory (Ben-Haim and Shanhav 1984, Ben-Haim 1985a).

This chapter resolves the separation of design and interpretation by proposing a multi-detector interpretation method. Multi-dimensional rays in conjunction with the response set are utilized for the purpose of developing a method that, given a set of detector responses, assesses the amount of activity.

Inferring a Detector Response from A Known Activity

Throughout the dissertation it is assumed that the detector responses, whether from a point source or some other distribution, are proportional to the amount of radioactivity present. Therefore, if a point $(f_1, f_2, \ldots, f_N)$ is the response from one unit of activity, then $(mf_1, mf_2, \ldots, mf_N)$ is the response from m units of activity.

One result of this property of proportionality is that all points $mf=(mf_1, mf_2, \ldots, mf_N)$ that are the result of some m units of activity from a specified distribution are located on

the same ray from the origin with a distance from the origin proportional to the m units of activity present. A point $(f_1, f_2, \ldots, f_N)$ which is the result of a unit amount of source activity is located on the ray intersecting the origin with the slopes

$$s_1 = \frac{f_2}{f_1},\ s_2 = \frac{f_3}{f_1},\ \ldots,\ s_{N-1} = \frac{f_N}{f_1}. \qquad (3\text{-}1)$$

The distance from the origin of this point is

$$d = \sqrt{f_1^2 + f_2^2 + \ldots + f_n^2}. \qquad (3\text{-}2)$$

If the amount of source is increased to m units, the detector system response is $(mf_1, mf_2, \ldots, mf_N)$. The slopes of this ray are identical to the slopes of the ray from one unit of activity. As shown in Figure 3-1 the distance from the origin of this point is

$$d = \sqrt{(mf_1)^2 + (mf_2)^2 + \ldots + (mf_n)^2} \qquad (3\text{-}3)$$

or m times the amount from one unit of activity.

Interpreting Activity from a Known Response Point

The above use of either method demonstrates that, given a detector response point from a known amount of activity, a response point from another amount of activity can be inferred. The problem of interpretation is to calculate the activity that results in a response point of unknown activity,

but identical source configuration, utilizing a response point of known activity. Since the points of known activity and unknown activity are located on the same ray and the distances from the origin are proportional to the amount of activity present, the unknown amount of activity is equal to the distance from the point response of the unknown activity divided by the distance from the response point of the known activity. Given a response point $(g_1, g_2, \ldots, g_N)$ from m_g units of activity and a response point from an unknown amount of activity $m_h = (h_1, h_2, \ldots, h_N)$, the amount of activity m_h is

$$m_h = m_g \frac{\sqrt{h_1^2 + h_2^2 + \ldots + h_N^2}}{\sqrt{g_1^2 + g_2^2 + \ldots + g_N^2}}. \tag{3-4}$$

A two-dimensional example is shown in Figure 3-2.

Interpretation of Responses from a Unit Response Point

For a given amount of activity, a response set may possess a range of possible responses along a ray. If the activity of a given detector response is inferred from a range of possible responses, the interpretation of the activity from the responses is performed by calculating two ratios. These ratios, in both cases, have as the numerator the distance from the origin of the response of unknown activity $(h_1, h_2, \ldots, h_N)$. The denominators are the maximum and minimum distances from the origin to the responses of known activity, where the distances are taken along the same ray that connects

$(h_1, h_2, \ldots, h_N)$ to the origin. The range of activities possible from the given response $(h_1, h_2, \ldots, h_N)$ is calculated by multiplying the respective distance ratios by the known activity, m_0, or

$$m_0 \frac{d_h}{d_g} \le m_h \le m_0 \frac{d_h}{d_f}. \tag{3-5}$$

Figure 3-3 illustrates the case where the known activity is a unit amount.

In order extend the above discussion to a complete response set, the line segment resulting from the intersection of the complete response set C and the ray defined by the given detector response $(h_1, h_2, \ldots, h_N)$ is calculated. The endpoints of this line segment determine the divisors for the ratios. The maximized and minimized ratios give the maximum and minimum activity that could result from the response point. If s_i are the slopes from the origin to the given detector response $(h_1, h_2, \ldots, h_N)$, defined by

$$s_1 = \frac{h_2}{h_1},\ s_1 = \frac{h_3}{h_1},\ \ldots,\ s_{N-1} = \frac{h_N}{h_1}, \tag{3-6}$$

the maximum and minimum amounts of activity are

$$\max_{c \in C} \frac{\sqrt{h_1^2 + h_2^2 + \ldots + h_N^2}}{\sqrt{c_1^2 + c_2^2 + \ldots + c_N^2}} \text{ and } \min_{c \in C} \frac{\sqrt{h_1^2 + h_2^2 + \ldots + h_N^2}}{\sqrt{c_1^2 + c_2^2 + \ldots + c_N^2}} \tag{3-7}$$

where for all c, $c_N = s_{N-1}c_1$, $c_{N-1} = s_{N-2}c_1$, $\ldots$, $c_2 = s_1 c_1$.

A ratio of the maximum over the minimum amount of activity is

written as

$$\max_{g \in c}\left(\frac{g_n}{f_n}\right), \text{ for any } n \qquad (3\text{-}8)$$

where for g and f, $f_N = s_{N-1}f_1$, $f_{N-1} = s_{N-2}f_1$, $\cdots$, $f_2 = s_1 f_1$

The point source response set can be formulated as the parametric representation of the spatial coordinates of a unit point source with a specified source container. The interpretation is performed over the complete response set which is the set of all convex combinations of the point source response set. The properties of the point source response set often allow the complete response set to be constructed with less complexity than resorting to calculating all the convex combinations in the point source set. The following examples illustrate the application of the line intersection method for a number of simplified detector and source configurations.

Example 3-1. The radioactivity is confined to the line segment $-0.5 \le x \le 0.5$. The detectors are located on the x axis at $x=1$ and $x=-1$ with response functions

$$f_1 = \frac{1}{(1+x)^2} \text{ and } f_2 = \frac{1}{(1-x)^2} \qquad (3\text{-}9)$$

The point source response set, which is determined from the above two response functions, is illustrated in Figure 3-4. The construction of the complete response set sometimes can be determined from the behavior of the point response functions. The second derivative of f_2 with respect to f_1 is

$$\frac{d^2 f_2}{d f_1^2}=\frac{(1+x)^4}{(1-x)^4}\geq 0. \tag{3-10}$$

Therefore the detector response of f_2 in terms of f_1 is concave upward. A line segment that connects two points from the point source response set is contained in the complete response set. Such a line segment is above or on the point source response set defined by the two points. The line segment that connects the endpoints of the point response set is above any other line segment that can be created from two points in the point response set. The boundaries of the complete response set are formed from the point response set and the line segment that connects the two extreme points of the point response set (4,0.444 and 0.444,4). The equation of the line that contains this line segment is $f_2=-f_1+4.444$. The complete response set is illustrated in Figure 3-5. If the given detector response is $(f_1,f_2)=(4,2)$, the slope of the line intersecting this point is $s=0.5$. Since $f_2=sf_1$, where

$$s=\frac{f_2}{f_1}=\frac{(1+x)^2}{(1-x)^2}=0.5, \tag{3-11}$$

the minimum detector response for the above line intersecting the point source response set occurs at $x=-0.172$. The minimum detector response is therefore (1.46,0.73). The maximum detector response occurs on the boundary of the complete response set but not on the point response set. The maximum detector response occurs at the intersection of $f_2=-f_1+4.444$

and $f_2=0.5f_1$, or $(f_1,f_2)=(2.96,1.48)$. As demonstrated in Figure 3-6, the maximum and minimum activity that could cause the response (4,2) is 2.74 and 1.35, respectively. The ratio of the maximum activity to minimum activity is 2.027.

The ratio of the maximum to minimum activity possible is dependent on the detector response that is initially obtained. For the above example, if the given detector response was $(f_1,f_2)=(0.444,4)$, the ratio of maximum to minimum activity is one. The maximum possible value of the ratio is 2.22 which is the result of the given response of detector #1 equal to the response of detector #2. This maximized ratio is equivalent to the RMR discussed previously. The utilization of the line intersection method for the RMR is discussed in the next chapter.

The above example can be extended into areas such as radioactive waste assessment. An applicable problem is the assessment of activity of a 55 gallon waste container. Waste container assays are usually performed by assuming that the activity is homogeneously distributed or that any heterogeneous distributions will not significantly affect the assessment. If it is no longer assumed that the activity possesses spatial homogeneity, the ray method allows for the use of multiple detector measurements to reduce the assessment uncertainty.

Example 3-2. The radioactivity is confined to an infinitesimally thin disk of radius R. As illustrated in

Figure 3-7, two detector readings are performed above the centerline of this disk. The problem is to determine the range of possible activities that could result in the two given detector readings.

Using cylindrical coordinates, the equation of the disk that confines the radiation is $r=R$. The detector readings are performed at $z=Z_1$ and $z=Z_2$ with $Z_2>Z_1$. The response functions are

$$f_1=\frac{1}{(Z_1^2+r^2)} \text{ and } f_2=\frac{1}{(Z_2^2+r^2)}. \tag{3-12}$$

The second derivative of f_2 with respect to f_1 is

$$\frac{d^2f_2}{df_1^2}=\frac{-2}{\left(f_1+(Z_2^2-Z_1^2)\right)^3}<0 \tag{3-13}$$

Therefore, the point response function of f_2 in terms of f_1 is concave downwards and the upper boundary of the complete response set is the point response set and the lower boundary of the complete response set is the line segment connecting the endpoints of the point response set. As shown in Figure 3-8, the boundaries of the complete response set are the point response set and the line segment

$$\begin{aligned}(c_1,c_2)&=(a f_1(0)+(1-a)f_1(1), a f_2(0)+(1-a)f_2(1))\\&=\left(\frac{a}{Z_1^2}+\frac{(1-a)}{Z_1^2+1}, \frac{a}{Z_2^2}+\frac{(1-a)}{Z_2^2+1}\right) \text{ where } 0\le a\le 1.\end{aligned} \tag{3-14}$$

The equation of the line containing this segment is

$$f_2 = wf_1 + b$$

$$\text{where } w = \frac{Z_1^2(Z_1^2+1)}{Z_2^2(Z_2^2+1)} \text{ and } b = \frac{Z_2^2-Z_1^2}{Z_2^2(Z_2^2+1)} \tag{3-15}$$

The ray from the origin to the point (h_1,h_2) intersects the line segment at

$$f_1 = \frac{bh_1}{h_2-wh_1},\ f_2 = \frac{bh_2}{h_2-wh_1} \tag{3-16}$$

Therefore f_1 and f_2 in terms of Z_1 and Z_2 are

$$f_1 = \frac{h_1(Z_1^2-Z_2^2)}{h_2Z_2^2(Z_2^2+1)-h_1Z_1^2(Z_1^2+1)},\ f_2 = \frac{h_2(Z_1^2-Z_2^2)}{h_2Z_2^2(Z_2^2+1)-h_1Z_1^2(Z_1^2+1)} \tag{3-17}$$

A ray from the origin to the point (h_1,h_2) intersects the point source response set at

$$f_1 = \frac{h_1-h_2}{h_2(Z_2^2-Z_1^2)},\ f_2 = \frac{h_1-h_2}{h_1(Z_2^2-Z_1^2)}. \tag{3-18}$$

As a numerical example, the two detectors readings are taken at Z_1=1 and Z_2=1.414. The detector responses are given as (h_1,h_2)=(1,0.55). The intersection with the point response set is (0.818,0.45). As demonstrated in Figure 3-9, the intersection with the line segment boundary of the complete response is (f_1,f_2)=(0.771,0.424). The range of activities that could result from the given detector response point (1,0.55) is 1.222 to 1.297 units of activity, a difference of 6.1%. If only one detector was available then the range of activities

possible from a detector reading of $h_1=1$ is one to two units of activity, a difference of 100%.

The above example serves as a simplified version of a number of applied problems. An unknown distribution of radioactive material on a sample planchet could be assessed by counting the planchet on two different shelf locations. An aerial survey could assess the amount of ground contamination by taking measurements at two different altitudes. Floor contamination of a room may be assessed by survey data taken at different heights above the floor.

Example 3-3. The above two detector problems have the luxury of the use of graphical techniques for visualization and calculation. For greater than two detectors the problems become more complex due to increasing number of constraints that are involved. A three detector problem utilizing the ray method is presented in the following example. The source is confined to a cube of dimension one by one by one centered on the origin. The radioactivity is restricted to a single point source. Radiation detectors #1, #2 and #3 are located respectively at (1,0,0), (-1,0,0) and (0,1,0). The three point source response functions in terms of the spatial x, y and z coordinates are defined as

$$f_1=\frac{1}{(1-x)^2+y^2+z^2},\ f_2=\frac{1}{(1+x)^2+y^2+z^2}\ \text{and}\ f_3=\frac{1}{x^2+(1-y)^2+z^2}$$

A given line passing through the origin is $f_1=s_1f_2$ and $f_1=s_2f_3$ or

$$\frac{1}{(1-x)^2+y^2+z^2}=\frac{s_1}{(1+x)^2+y^2+z^2} \text{ and } \frac{1}{(1-x)^2+y^2+z^2}=\frac{s_2}{x^2+(1-y)^2+z^2}$$
$$\text{where } -.5\le x,y,z\le .5$$

The above constraints define the line segment resulting from the intersection of a ray and the point source response set. This line segment contained within the convex set is written as two equations with y and z in terms of x,

$$y=bx,\quad z^2=-x^2(1+b^2)+2x\frac{(s_1+1)}{(s_1-1)}-1$$
$$\text{where } -.5\le x,y,z\le .5 \text{ and } b=\frac{-2s_2+s_1+1}{s_1-1} \tag{3-21}$$

Given an s_1 and s_2, the above equation defines a curve in (x,y,z) space dependent on x (with y linear and z parabolic). The curve, however, may not be valid in the entire domain of x since the ranges of y and z need to be satisfied. Combining the above functions with the inequalities gives the possible domain of x in terms of s_1 and s_2 as

$$x=.\frac{5}{b}y \text{ and } x=w\pm\frac{\sqrt{w^2-(b^2+1)(z^2+1)}}{b^2+1}$$
$$\text{where } w=\frac{s_1+1}{s_1-1},\ b=\frac{2s_2+s_1+1}{s_1-1},\ -.5\le y,z,\le .5. \tag{3-22}$$

The interpretation equation for a given (k_1,k_2,k_3) is

$$\min_{g\in C}\frac{\sqrt{h_1^2+h_2^2+\ldots h_N^2}}{\sqrt{g_1^2+g_2^2+\ldots+g_N^2}}\le m_3\le \max_{f\in C}\frac{\sqrt{h_1^2+h_2^2+\ldots+h_N^2}}{\sqrt{f_1^2+f_2^2+\ldots+f_N^2}} \tag{3-23}$$

where for g and f: $f_n = s_{N-1} f_1$, $f_{N-2} = s_{N-1} f_1, \ldots, f_2 = s_1 f_1$ (3-24)

After substituting the constraints the interpretation is written as

$$\min_z \frac{h_1}{g_1(x,y,z)} \le m_h \le \max_z \frac{h_1}{g_1(x,y,z)} \tag{3-25}$$

Upon substituting the appropriate functions for y and z, $g_1(x,y,z)$ is written as

$$g_1(x,y,x) = \frac{1}{(1-x)^2 + b^2x^2 - x^2(1+b^2) + 2wx - 1} = \frac{1}{2x(w-1)} \tag{3-26}$$

The amount of activity present is written as

$$\min_x(k_1(2x(w-1)) \le m_h \le \max_x(2x(w-1)). \tag{3-27}$$

Two numerical examples are given. A detector response is given as $(h_1,h_2,h_3)=(8,0.888,1.6)$. The slopes are calculated as s_1=8/0.888=9 and s_2=8/1.6=5. The variables for the problem are b=0 and w=10/8=1.375. The value of b=0 gives y as zero and x as indeterminate. Using w=1.375 gives the range of x as

$$\min_z\left(\frac{10}{8} \pm \sqrt{\left(\frac{10}{8}\right)^2 - (z^2+1)}\right) \le x \le \max_z\left(\frac{10}{8} \pm \sqrt{\left(\frac{10}{8}\right)^2 - (z^2+1)}\right) \tag{3-28}$$

The value of x is restricted to only 0.5. For any other value of z besides zero, x is greater than 0.5. Therefore, the amount of activity resulting from the detector response $(h_1,h_2,h_3)=(8,0.888,1.6)$ is known exactly to be 8(1.25-1)=2 units of activity.

As a second numerical example, the detector responses are given as $(h_1,h_2,h_3)=(2.909,1.185,2.909)$. The constants for the problem are $s_1=2.455$, $s_2=1$, $b=1$ and $w=2.375$. The applicable range of x is based on the following equation

$$x=\frac{2.375-\sqrt{2.375^2-2(z^2+1)}}{2}, \text{ where } 0\le z^2\le 25 \qquad (3\text{-}29)$$

which allows x to possess a range of .233 to .301. The amount of activity present is

$$1.865\le m_s\le 2.409. \qquad (3\text{-}30)$$

In summary, the proposed N detector interpretation problem has an N vector response point and an N dimensional unit response set. The response point defines a ray from the origin composed of N-1 linear equations, which intersects the N dimensional unit response set. The distance of the given response point from the origin divided by the distance from the origin of the endpoints of the line segment intersecting the unit response set give the maximum and minimum amounts of activity that could result in the given response set. Due to the linear relationship between the detector responses, the response of only one detector is required for the calculation of the ratios. The N-1 linear equations allow for the formation of N-1 relationships between the spatial variables (x,y,z). These linear equations limit the permissible ranges of the spatial variables. In addition, these linear equations can reduce the

number of dependent spatial variables of the detector response function.

In order to determine the range of possible activity, a search procedure must determine the maximum and minimum detector response, subject to the constraints of the problem. In terms of the spatial variables, both the detector responses and the constraints are nonlinear functions. No general method exists for optimizing a nonlinear function subject to nonlinear constraints. Utilizing a local search procedure for the above type problem may result in a local optimum that is not the global optimum of the problem. In terms of detector space however, the search of the interpretation problem is constrained to a line segment. The endpoints of the line segment contain the maximized and minimized detector responses.

The interpretation problem is defined ultimately in terms of the spatial variables which do not linearly constrain the problem. In order to utilize the linear constraint of the problem, the following search procedure is recommended. After the detector response appears to be optimized in terms of the spatial variables, the optimized detector response is incremented in the direction (with respect to detector space) that the search has previously progressed. The spatial variables that are calculated from the incremented detector response are determined to be within or without the permissible ranges. If the variables satisfy the ranges, then

the calculated response is not an optimum solution. The incremented response is used as an initial point for restarting the search.

In order to include unconstrained distributions, the search is required to be performed over the complete response set. The search over the complete response set introduces additional variables. Examples 3-2 and 3-3 constructed the boundaries of the complete response set by determining the curvature of the point source response function. If the curvature were solely positive or negative, an upper of lower boundary is determined. A more general method of searching over the complete response set is performed by generating a detector response point from the complete response set from N points of the point response set in conjunction with N-1 independent coefficients.

A set of detector responses from a two detector point source response set is represented as (f_1, f_2). A point from the complete response set is represented as (c_1, c_2) or in terms of (f_1, f_2) as

$$c_1 = \alpha f_1 + (1-\alpha) f_1' \text{ and } c_2 = \alpha f_2 + (1-\alpha) f_2'. \qquad (3\text{-}31)$$

If a given detector response point (h_1, h_2) results in the slope s then

$$c_2 = \alpha f_2 + (1-\alpha) f_2' = \alpha s f_1 + (1-\alpha) s f_1' \qquad (3\text{-}32)$$

in terms of s and the points (f_1, f_2) and (f_1', f_2') from the point source response set, alpha is

$$\alpha = \frac{sf_1' - f_2'}{-sf_1 + f_2 + sf_1' - f_2'} \text{ where } 0 \le \alpha \le 1. \tag{3-33}$$

The interpretation equation is

$$\min_{f'}\left(\frac{h_1}{\alpha f_1 + (1-\alpha) f_1'}\right) \le m_h \le \max_{f'}\left(\frac{h_1}{\alpha f_1 + (1-\alpha) f_1'}\right) \tag{3-34}$$

The search of the optimum detector responses can, in some cases be simplified by the substitution of the appropriate extreme point for either f'. The appropriate point will allow the entire complete response set to be evaluated by the proper choice of variables for alpha and f. If f' can remain fixed, then the search for a two detector problem over the complete response set is reduced to a single variable.

Example 3-4. The radioactivity is confined to a segment on the x axis from $x=-0.5$ to $x=0.5$. The detectors are located on the x axis at $x=-1$ and $x=1$ with response functions

$$f_1 = \frac{1}{(1+x)^2} \text{ and } f_1 = \frac{1}{(1-x)^2} \tag{3-35}$$

A point in the complete response set is written as

$$(c_1, c_2) = \left(\frac{\alpha}{(1+x)^2} + \frac{(1-\alpha)}{(1+x')^2}, \frac{\alpha}{(1-x)^2} + \frac{(1-\alpha)}{(1-x')^2}\right). \tag{3-36}$$

If a given detector response results in the slope s the constraint is

$$c_2 = sc_1 = \frac{\alpha}{(1-x)^2} + \frac{(1-\alpha)}{(1-x')^2} = \frac{s\alpha}{(1+x)^2} + \frac{s(1-\alpha)}{(1+x')^2} \tag{3-37}$$

In terms of the two elements from the point source response

set, alpha is

$$\alpha=\frac{\frac{s}{(1+x')^2}-\frac{1}{(1-x')^2}}{\frac{1}{(1-x')^2}-\frac{s}{(1+x)^2}+\frac{s}{(1+x')^2}-\frac{1}{(1-x')}} \tag{3-38}$$

The search over x and x' is reduced to a search over x by recognizing that if the complete response set can be generated from a fixed extrema point on the point response set, such as $(f_1,f_2)=(.444,4)$, then a variable point on the point source response set and the line segment connecting the two points is sufficient to define the complete response set. If the variable point is $(f_1,f_2)=(4,.444)$, the line segment connecting this point with the fixed point $(.444,4)$ creates part of the boundary of the complete response set.

If the $x'=.5$, then alpha is

$$\alpha=\frac{1.778(.25s-2.25)}{\frac{-(s-1)+2x(s+1)-x^2(s-1)}{(1+x)^2(1-x)^2}+1.778(.25s-2.25)} \tag{3-39}$$

$$\text{where } 0\le\alpha\le1$$

An alpha outside the range of (0,1) indicates that the intersection of the line created from the variable and fixed response points and the ray created from the given response point of unknown activity occurs outside the line segment and therefore outside the complete response set.

If the detector response from an unknown amount of activity is given as $(h_1,h_2)=(2,4)$ $(s=2)$, alpha is

$$a=\frac{-3.111}{\frac{-1+6x-x^2}{(1+x)^2(1-x)^2}-3.111} \quad \text{where } 0\le a\le 1. \tag{3-40}$$

The range of the amount of activity that could result in the detector response (2,4) is written as the following constrained optimization

$$\min_x\left(\frac{2}{\frac{a}{(1+x)^2}+\frac{(1-a)}{(1+.5)^2}}\right)\le m_b\le \max_x\left(\frac{2}{\frac{a}{(1+x)^2}+\frac{(1-a)}{(1+.5)^2}}\right) \tag{3-41}$$

$$\text{subject to: } 0\le\frac{-3.111}{\frac{1}{(1-x)^2}-\frac{2}{(1+x)^2}-3.111}\le 1$$

Figure 3-10 illustrates the search for a fixed point (0.444,4) and a variable point (1,1) resulting in the point from the complete response set (0.87,1.72).

The constraint limits the range of x to between x=-0.5 (alpha=.29) and x=.17 (alpha=1). The amount of activity that could be present is

$$\left(\frac{2}{\frac{.29}{.5^2}+\frac{.71}{1.5^2}}\right)\le m_b\le\left(\frac{2}{\frac{1}{1.17^2}}\right) \tag{3-42}$$

or

$$1.35\le m_b\le 2.74 \tag{3-43}$$

giving the same result as found in example 3-1.

If the unit set of detector responses are generated from a computer code, the search procedure can not take advantage of any analytical relationships. Without the advantage of analytical relationships to simplify the search, the assessor

is confronted with the choice of generating elements of the complete response set from varying spatial distributions or generating only elements of the point source response set. If only the point source response set is generated, then the search over the complete response set is performed by either convex combinations of the point source response set or evaluating the linear N-1 dimensional hyperplanes which define the boundaries of the complete response set. If the boundaries of the complete response set are determined, the search is simplified to the point response set and the linear boundaries. If the search is performed by convex combinations, then the number of independent variables in the search is equal to the number of detectors present. Although the size of the generated set determines partially the accuracy and CPU time necessary to optimize the responses, the same consideration illustrated in the evaluation of the above simplified examples, that of exploiting the properties of the point response set in order to simplify the search of the complete response set, is an excellent tool for the reduction of the calculational effort.

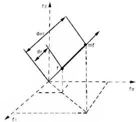

Figure 3-1. Relationship between a response from a unit activity and a response from a unit activity multiplied by a.

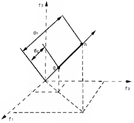

Figure 3-2. The amount of activity from point h that is inferred from point g is $m_g(d_h/d_g)$.

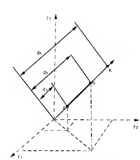

Figure 3-3. The amount of activity from point k that is inferred from the unit activity line segment which connects points g and h is $(d_k/d_h) \leq m_k \leq (d_k/d_g)$.

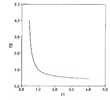

Figure 3-4. Point source response set for a line source container.

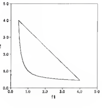

Figure 3-5. Complete response set for a line source container.

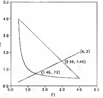

Figure 3-6. Line intersection method with a two detector complete response set.

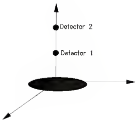

Figure 3-7. Detector configuration for disk container.

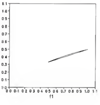

Figure 3-8. Complete response set for two detectors with disk source container.

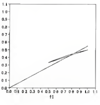

Figure 3-9. The line intersection method for the complete response set of two detectors with a disk source container

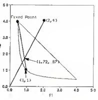

Figure 3-10. Fixed point search procedure with variable point at (1,1) and fixed point at (4,.444).

CHAPTER 4
RELATIVE MASS RESOLUTION

Introduction

The previous chapter expanded the application of dimensional space to include the interpretation of multiple detector responses by the utilization of a line intersection method. This method solves for the range of activity that could result from a given N detector response. Different N detector responses result in a different ranges of interpreted activity. To quantify this range, the ratio of the maximum to minimum amount of activity of this range is calculated.

The application of dimensional space has been previously applied in published work to the problem assessing the maximum uncertainty of a detector system. This uncertainty is determined by the calculation of the RMR which has been defined as the ratio $(m+\Delta)/m$ where $(m+\Delta)$ is the smallest quantity of radioactive material greater than an amount m of radioactive material such that any spatial distribution of more than $(m+\Delta)$ is distinguishable from any spatial distribution of m units (Ben-Haim 1983). For an N detector system, the use of the RMR provides a single measurement that aids in assessing the performance of the detector system. Such

a performance criterion could assist both the analyst and the designer detector systems.

A previously developed algorithm (Ben-Hais 1983), which utilizes supporting planes, determines the RMR by searching over the point source response set versus the complete response set, thus reducing the number of variables required for the search. The method is restricted to the case where the response set is convex. Since the complete response set is the convex hull of the point source response set, the restriction does not pose a barrier to the solution of the RMR for unconstrained source distributions. If the possible set of source distributions is restricted, however, then the resulting response set might not be convex. A source restricted to a single point source results in a response set which, in general, is not a convex set. In addition, the above search procedure does not guarantee that a local optimum solution is a global optimum solution.

In order to address the above shortcomings, the line intersection method of the previous chapter is extended to include a method for searching for the RMR. The ray method, in addition to incorporating the same formulations that were used in interpreting a detector response, is able to search for the RMR over nonconvex response sets and guarantee that a local optimum is a global optimum for convex response sets.

The number of independent variables required for the search of the RMR is dependent on the number of measurements.

If relationships can be developed between the slopes of the upper and lower supporting planes and the response set, the number of independent variables can be reduced. A method of utilizing the derivatives of the point source response set has been developed which can reduce both the number of variable and the number of required optimization steps.

Unlike the problems presented in this dissertation, the calculation of most multiple detector problems will require computer based search methods. If local type search methods are utilized, local optimum are not necessarily the global optima of the solution. Therefore the development of methods that guarantee the global applicability of local solutions have been investigated. The line intersection method for the RMR is demonstrated to be a global optimization method for convex sets. Finally, a technique has been introduced into the supporting plane search procedure to allow for the global optimization of convex sets.

Basic Formulation of the RMR

Distinguishing Activity Along a Ray

The description of the RMR has been previously formulated in terms of N-1 dimensional hyperplanes. The RMR, however, can be described in terms of distances from the origin. The following discussion of the RMR dovetails two fundamental descriptions of the RMR: the N-1 dimensional hyperplane approach from previous literature and the distance from the

origin approach developed in the previous chapter. The addition of the distance to the origin approach allows for the solution of problems which the method of hyperplanes does not address. Before discussing multi-detector formulations and the techniques developed, the following single detector example problem illustrates that, although the concept of the RMR appears foreign to practitioners of waste assay, the RMR, at least for the one dimensional response case, is an intuitive concept for measuring the performance of a detector configuration.

Example 4-1. The radioactivity is confined to the line segment $-5 \le x \le 5$. The detector is located on the x axis at $x=1$ with response function

$$f=\frac{1}{(1-x)^2} \tag{4-44}$$

The detector response to a point source of unit activity is from 0.44 to 4 inclusive. If the unit point source is divided into a number of subunit point sources, any possible detector response resulting from a distribution of these subunit sources will be between 0.44 and 4. Assuming proportionality of detector response to activity, nine units of activity will result in a response between 4 and 36. If the detector response for some unknown amount and distribution of activity is given as four, then the possible amount of activity present is between one unit and nine units. Therefore, the RMR of this single detector system is nine over one or nine. However,

instead of calculating the RMR from its formal definition $(m+\Delta)/m$, the RMR can be determined by calculating the ratio of the maximum and minimum responses from a set of responses with a fixed activity. In the above example, the RMR can be calculated from only the unit response set. The maximum over the minimum response for one unit is 4/0.44 or 9.

Since a single unit source will result in more extreme single detector responses than a divided unit source, the ratio of the maximum to minimum detector response from a point source, at least for a single detector response with the assumption of proportionality of response to activity, determines the RMR. Part of the problem in extending the above example to multiple measurements is the confusion as to the representation of an extreme response and how point source distributions can "contain" other distributions.

The application of dimensional space for multiple detector responses allows the definition of a maximum and a minimum detector response to be extended to multiple detector configurations. The criterion that determines the magnitude of a multiple detector response is the distance of a multiple detector response from the origin in response space. The application of the complete response and point source response set allows for the relationship between distributions of subunit sources and unit point sources to be extended to multiple detector configurations.

The line intersection method of the previous chapter considered a line segment resulting from a ray intersecting the complete response set. Such a segment represents different detector responses due to various spatial distributions of a fixed amount of activity. In addition, the line segment represents proportional detector responses which are the result of fixed spatial distributions of various proportional amounts of activity. For example, a detector response twice the distance from the origin of another detector response is the result of twice as much activity. Therefore, the set of detector responses along the line segment are the result of two different means: different distributions with fixed amounts of activity and fixed distributions with different amounts of activity.

The calculation of the RMR along a segment of a ray from the origin can be considered a one dimensional detector problem. The distances of the endpoints from the origin of the line segment are the maximum and minimum responses. The RMR is calculated by dividing the distance of the endpoint that is further from the origin by the distance that is closer to the origin. In terms of distances from the origin, the equation of the RMR is

$$\mathrm{RMR} = \frac{\max\sqrt{h_1^2+h_2^2+\cdots+h_M^2}}{\min\sqrt{g_1^2+g_2^2+\cdots+g_M^2}} \qquad (4\text{-}45)$$

As mentioned before, supporting planes can be utilized to determine the RMR. A result of the property of proportionality

of response to activity is that all points $(mf_1, mf_2, \ldots, mf_N)$ from an arbitrary amount of activity m with a specific source distribution can be intersected by N-1 dimensional hyperplanes that are a distance from the origin proportional to the amount of activity present. For example, a point $(g_1, g_2, \ldots, g_N)$ from a unit amount of activity is intersected by a plane $b{=}a_1f_1{+}a_2f_2{+}\ldots{+}a_Nf_N$, where b is satisfied by $b{=}a_1g_1{+}a_2g_2{+}\ldots{+}a_Ng_N$. The perpendicular distance from the origin to this plane is

$$d=\frac{|b|}{\sqrt{a_1^2+a_2^2+\ldots+a_N^2}}=\frac{|b|}{|a|} \tag{4-46}$$

The equation of a parallel hyperplane that intersects the point $(mg_1, mg_2, \ldots, mg_N)$ is $\langle a, f\rangle = b_1$ where

$$b_1=a_1mg_1+a_2mg_2+\ldots+a_Nmg_N \tag{4-47}$$

The perpendicular distance from this plane to the origin is

$$d_1=\frac{|b_1|}{\sqrt{a_1^2+a_2^2+\ldots+a_N^2}}=\frac{m|b|}{\sqrt{a_1^2+a_2^2+\ldots a_N^2}}=\frac{m|b|}{|a|} \tag{4-48}$$

which is m times the activity of a unit response as illustrated in Figure 4-1. The ratio of the above distances is written as

$$\frac{d_1}{d}=\frac{|b_1|}{|a|}\frac{|a|}{|b|}=\frac{|b_1|}{|b|}=\frac{\langle a, mg\rangle}{\langle a, g\rangle} \tag{4-49}$$

The RMR along the line segment is the ratio of the maximum to minimum distance from the origin or

$$\mathrm{RMR}=\frac{d_1}{d}=\frac{|b_1|}{|b|}=\frac{\max_h|\langle a, h\rangle|}{\min_g|\langle a, g\rangle|} \tag{4-50}$$

Since the detector responses are confined to a line segment contained in the ray having the equation

$$f_1=s_1f_2,\ f_1=s_2f_3,\ \ldots,\ f_1=s_{N-1}f_N, \tag{4-51}$$

the resulting equation for the RMR of either the supporting plane method or the distance from the origin method, simplifies to

$$\mathrm{RMR}=\max_{g,h}\left(\frac{g_n}{h_n}\right) \text{ for any } n. \tag{4-52}$$

Distinguishing Activity within a Set

The calculation of the RMR is usually not confined to a single segment of a ray from the origin but to a set of responses, such as the complete response set. The complexity of the search for the RMR over a set versus over a ray increases, since the RMR calculated for one particular line segment may be different than the RMR calculated for another line segment. If a response set is multiplied by a ratio that is less than the RMR of the set, the multiplied set will contain points in common with the initial set. The RMR can be calculated by searching for the line segment which results in the greatest maximum to minimum ratio. If the calculation of the RMR is formulated in terms of distances from the origin, the procedure of finding the RMR over a complete set is similar to the ray method outlined in Chapter 3 with the addition that the calculation is performed over all possible rays from the origin that intersect the response set, C. The ray method formulation of the RMR, over a set C, is

$$\mathrm{RMR}=\max_{a}\frac{\max_{h\in C}\sqrt{h_1^2+h_2^2+\ldots+h_n^2}}{\min_{g\in C}\sqrt{g_1^2+g_2^2+\ldots+g_n^2}}\quad \text{for any } n, \tag{4-53}$$

where for g,h: $f_1=a_1f_2$, $f_2=a_2f_3$, ..., $f_{n-1}=a_{n-1}f_n$

If the response set is convex, the method of supporting hyperplanes can be utilized to determine the RMR. The distances from the origin of the upper and lower supporting planes for given vector $a=(a_1,a_2,\ldots,a_n)$ and a set C are calculated, as illustrated in figure 4-2. The ratio of the distances of these parallel planes serves as the trial factor, m_T, of the RMR. The set calculated when each point in C is multiplied by the trial factor m_T is m_TC. The response points m_TC are located on upper supporting plane of C and the lower supporting plane of m_TC. The sets C and m_TC are separated by the upper supporting plane of C (or the lower supporting plane of m_TC). The only location for identical response points between the sets C and m_TC is on the upper supporting plane of the set C. If a point from the set C is multiplied by m_T, the point m_TC is located on or above the upper supporting plane. If the sets m_TC and C do not have points in common along the upper supporting plane, then the RMR is smaller than the trial factor. Due to the property that convex sets contain all convex combinations of their elements, the smallest trial factor from the different possible slopes of the supporting planes is the RMR (Ben-Haim 1985a). The equation for the RMR for a convex set C is

$$\mathrm{RMR}=\min_{a_1,\ldots,a_N}\frac{\max_{h\in C}(a_1h_1+a_2h_2+\ldots+a_Nh_N)}{\min_{g\in C}(a_1g_1+a_2g_2+\ldots+a_Ng_N)} \qquad (4\text{-}54)$$

The correct slope of the supporting planes of a given convex set C is illustrated in Figure 4-3. If the set C is not convex, then the above equation does guarantee the solution to the RMR.

If the response set C is the convex hull of a set F, the search for the supporting planes can be performed over the set F versus the set C. The elements g and h in the above equation are then required to belong to F instead of C. Since the point source response set is easier to formulate than the complete response set, the method of supporting planes is less complex than the method utilizing rays intersecting the response set.

An algorithm based on the utilization of supporting planes for the evaluation of the RMR has been considered as an inner and outer search (Ben-Haim 1983). The inner search locates the upper and lower supporting planes for a given $(a_1,a_2,\ldots,a_N)$ by maximizing and minimizing the numerator and denominator of the above equation. The upper supporting plane of the set F is also the upper supporting plane of the set C, while $\min_g<a,g>$ is the lower supporting plane of the set C. The minimization of the ratio, $\max_h<a,h>/\min_g<a,g>$, over a is the "outer" search. The number of independent variable in the total search is $3N$-1 where N is the number of detectors present.

A number of difficulties are present in the application of the algorithm. While a gradient type optimization has been recommended as a search method, both the "inner" and "outer" searches suffer from the problem that a local optimum is not necessarily of global optimum (Scott 1991). In addition, in order to prevent the algorithm from resulting in solution of a set of planes that "straddle" the origin, a rejection criterion is established that an optimum solution must have agreement between the signs of the numerator and denominator (Ben-Haim 1983, pg. 58). No follow-up procedure is given, however, if an optimized solution must be discarded for nonagreement of signs (Scott 1991). In addition, the search, if implemented as described, could result in zero denominators (Scott 1991).

The supporting plane search method for the BMR, although not a global optimization technique, possesses the advantage of having the search space limited to the point source response set (Ben-Haim 1985b). The search is performed over a smaller search space than the ray method. The result of the smaller search space is that the maximization and minimization of the respective numerator and denominator is performed over only one point and N-1 weight coefficients for the supporting plane method versus N points, N-1 weight coefficients and the slope of the rays that intersect the response set.

Use of Derivatives as Weight Coefficients

If a mathematical relation between the "inner" and "outer" search is constructed, the number of independent variables and the number of optimization steps can be reduced. If the point source response function is a convex function, all linear combinations of the point source response set will be greater than or equal to the point source response set. If the derivative exists where the lower supporting plane intersects the response set, the slope coefficients of a lower supporting plane can be written in terms of the derivatives of the tangent point. Since the upper supporting plane is parallel to the lower supporting plane, the slopes upper supporting plane can be defined in terms of the derivatives of the point source response set. Starting from the equation for the RMR

$$\mathrm{RMR}=\min_a\left(\frac{\max_h(a_1h_1+a_2h_2+\ldots+a_Nh_N)}{\min_g(a_1g_1+a_2g_2+\ldots+a_Ng_N)}\right), \qquad (4\text{-}55)$$

if the numerator and denominator of the above equation is divided by a_N, the resulting equation is

$$\mathrm{RMR}=\min_a\left(\frac{\max_h\left(\frac{a_1}{a_N}h_1+\frac{a_2}{a_N}h_2+\ldots+\frac{a_N}{a_N}h_N\right)}{\min_g\left(\frac{a_1}{a_N}g_1+\frac{a_2}{a_N}g_2+\ldots+\frac{a_N}{a_N}g_N\right)}\right) \qquad (4\text{-}56)$$

The directional derivatives, a_m/a_N of the plane can be written in terms of the derivatives of the tangent point as

$$\frac{a_K}{a_N}=-\frac{df_N}{df_K} \text{ and } \frac{a_H}{a_N}=\frac{df_H}{df_N} \tag{4-57}$$

If the above relation is substituted for the coefficients of the equations for the supporting plane, the equation for the RMR is

$$\text{RMR}=\min_a\left(\frac{\max_h\left(-\frac{dg_N}{dg_1}h_1-\frac{dg_N}{dg_2}h_2-\ldots+\frac{dg_N}{dg_N}h_N\right)}{\min_g\left(-\frac{dg_N}{dg_2}g_1-\frac{dg_N}{dg_2}g_2-\ldots+\frac{dg_N}{dg_N}g_N\right)}\right) \tag{4-58}$$

The following examples illustrate the reduction of variables and optimization steps possible from utilizing the derivatives of the point source response set.

Example 4-2. The radioactivity is confined to the line segment $-5\leq x\leq 5$. The detectors are located on the x-axis at $x=R_1$ and $x=-R_2$. The point source response set F is composed of all detector responses of the form

$$f_1=\frac{1}{(R_1-x)^2} \text{ and } f_2=\frac{1}{(R_2+x)^2}. \tag{4-59}$$

The optimization equation for the RMR is

$$\text{RMR}=\min_{a_1\ a_2}\max_{g\ h\in F}\left(\frac{a_1h_1+a_2h_2}{a_1g_1+a_2g_2}\right) \tag{4-60}$$

Three optimization steps are required with the optimization of g and h nested in the optimization of $a(a_1,a_2)$. By understanding the curvature of the point source response set, a relationship between the optimization of the "inner" and "outer" searches is formulated. The derivative is

$$\frac{df_2}{df_1}=-\frac{(R_1-x)^3}{(R_2+x)^3} \tag{4-61}$$

Since the second derivative

$$\frac{d^2 f_2}{d f_1^2} = \frac{(R_1-x)^4}{(R_2+x)^4} \tag{4-62}$$

is greater than zero, the function is concave upwards and the slope of the lower supporting plane is defined by the point source response function and the slope of the upper supporting plane is identical to the slope of the lower supporting plane. The equation for the RMR in terms of the derivative of the point source response function is derived as follows:

$$\mathrm{RMR} = \max_{g,h} \left(\frac{-\frac{dg_2}{dg_1} h_1 + h_2}{-\frac{dg_2}{dg_1} g_1 + g_2} \right) \tag{4-63}$$

$$= \max_{g,h} \left(\frac{\left(\frac{R_1-x}{R_2+x}\right)^3 h_1 + h_2}{\left(\frac{R_1-x}{R_2+x}\right)^3 \frac{1}{(R_1-x)} + \frac{1}{(R_2+x^3)}} \right) \tag{4-64}$$

$$= \max_{x,h} \left(\frac{\left(\frac{R_1-x}{R_2+x}\right)^3 h_1 + h_2}{\left(\frac{R_1+R_2}{R_2+x}\right)^3} \right) \tag{4-65}$$

$$= \mathrm{RMR} = \max_{x,h} \left(\frac{(R_1-x)^3 h_1 + (R_2+x)^3 h_2}{(R_1+R_2)} \right). \tag{4-66}$$

Setting the derivative of the RMR with respect to x equal to zero results in

$$\frac{d(\mathrm{RMR})}{dx} = \frac{-2(R_1-x)^2 f_{1max} + 2(R_2+x)^2 f_{2max}}{R_1+R_2} = 0 \tag{4-67}$$

and

$$\frac{(R_1-x)^2}{(R_2+x)^2} = \frac{f_{2max}}{f_{1max}} \tag{4-68}$$

The above solution, which minimizes the RMR, requires that the minimized response point be identical to the maximized

response point. Therefore, an unconstrained optimization approach is not feasible and the constraints of the slopes of the supporting lines are required to be considered. Since the point source response set has two extreme points (f_{1max}, f_{2min}) and (f_{1min}, f_{2max}) which limit the range of possible slopes of the supporting lines, the search over the derivative df_2/df_1 is constrained. The line connecting these two points is part of the convex hull of the complete response set. If the slope of this line is s the derivative df_2/df_1 is constrained by this value of s, which is written as

$$\frac{df_2}{df_1}=s=-\frac{(R_1-x)^3}{(R_1+x)^3} \qquad (4\text{-}69)$$

The value of x at the constraint is

$$x=\frac{(R_1-R_2\sqrt[3]{s})}{(1+\sqrt[3]{s})} \qquad (4\text{-}70)$$

Substituting the above value for x into the equation for the RMR gives

$$\mathrm{RMR}=\max\left(\frac{\left(R_1-\frac{(R_1-R_1\sqrt[3]{s})}{(1+\sqrt[3]{s})}\right)^3 f_1+\left(R_1+\frac{(R_1-R_1\sqrt[3]{s})}{(1+\sqrt[3]{s})}\right)^3 f_2}{R_1+R_2}\right). \qquad (4\text{-}71)$$

The search is performed by checking the extreme points of the point source response set. If $R_1=R_2$ and $s=1$, the equation of the RMR is

$$=\max_f\frac{(f_1+f_2)}{2}=\max_x\frac{R^2+x^2}{(R-x)^2(R+x)^2} \qquad (4\text{-}72)$$

which is a maximum at $x=.5$ or $x=-.5$, resulting in an RMR of 2.22.

Example 4-3. Consider two radiation readings taken at Z_1=1 and Z_2=1.414 above the centerline of a thin disk with response functions

$$f_1=\frac{1}{1+r^2} \text{ and } f_2=\frac{1}{2+r^2} \qquad (4\text{-}73)$$

If f_1 is solved for r and substituted into the function for f_2 then

$$f_2=\frac{f_1}{f_1+1} \text{ and } \frac{df_2}{df_1}=\frac{1}{(f_1+1)^2}=\frac{1}{\left(\frac{1}{(r^2+1)}+1\right)^2} \qquad (4\text{-}74)$$

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Simplifying the expression for the derivative results in

$$\frac{df_2}{df_1}=\left(\frac{(r^2+1)}{(r^2+2)}\right)^2. \qquad (4\text{-}75)$$

Since

$$\frac{d^2f_2}{df_1^2}=-\frac{2}{(f_1+1)^3}<0, \qquad (4\text{-}76)$$

the point source response function is concave downward and tangent to the upper supporting plane. The RMR is written as

$$\text{RMR}=\max_{f,r}\left(\frac{-\left(\frac{r^2+1}{r^2+2}\right)^2\left(\frac{1}{r^2+1}\right)+\left(\frac{1}{r^2+2}\right)}{-\left(\frac{r^2+1}{r^2+2}\right)^2 f_1+f_2}\right). \qquad (4\text{-}77)$$

The use of derivatives has eliminated both the "inner" search and the slope coefficients of the supporting lines. Simplifying the above equation gives

$$\text{RMR}=\max_{f,r}\left(\frac{1}{-(r^2+1)f_1+(r^2+2)f_2}\right). \qquad (4\text{-}78)$$

The derivative is constrained to

$$\frac{df_2}{df_1}=\frac{f_2(1)-f_2(0)}{f_1(1)-f_1(0)}=\frac{\frac{1}{3}-\frac{1}{2}}{\frac{1}{2}-1}=\frac{1}{3} \qquad (4\text{-}79)$$

The value of r^2 at the above constraint is

$$r^2=\frac{2\sqrt{3}-1}{1-\sqrt{3}}=\frac{2\frac{\sqrt{3}}{3}-1}{1-\frac{\sqrt{3}}{3}}=0.366. \qquad (4\text{-}80)$$

Substituting .366 for r^2, results in $(f_1,f_2)=(1,.5)$ and an RMR of 1.072.

Utilizing derivative information for cases where the point source response set is convex or concave allows for an elimination of the slope coefficients of both the supporting hyperplanes and the "inner" search. If the point source response set is not strictly convex or concave the mathematical relationships formed from an investigation of the derivatives could still allow for a reduction in the number of local search variables required.

Use of the Ray Method for Nonconvex Sets

The supporting plane method only searches for the RMR for a convex set (Ben-Haim 1983). If the set of spatial distributions is constrained, the resulting response set may not be convex. A common constraint is to restrict the activity to a single point source. The formulation for the RMR over a point source response set is

$$\mathrm{RMR} = \max_s \max_{y,h} \frac{\sqrt{g_1^2+g_2^2+\cdots+g_N^2}}{\sqrt{f_1^2+f_2^2+\cdots+f_N^2}} \tag{4-81}$$

where for g and f: $f_N = s_{N-1} f_1$, $f_{N-2} = s_{N-1} f_1$, $\cdots$, $f_2 = s_1 f_2$ and s_n are the slopes to some given detector response $(h_1, h_2, \ldots, h_N)$.

Example 4-4. The following two detector configuration restricts the distribution of activity to a single point source which is confined it to a square area centered on the origin. The dimension of the square is one by one with the sides parallel to the x and y axis. Detectors #1 and #2 are located at point (1,0) and (-1,0) respectively. The point source response functions are

$$f_1 = \frac{1}{(1-x)^2+y^2} \text{ and } f_2 = \frac{1}{(1+x)^2+y^2} \tag{4-82}$$

The RMR is

$$\mathrm{RMR} = \max_s \max_{x,y} \frac{(1-x_1)^2+y_1^2}{(1-x_2)^2+y^2}$$

$$\text{subject to } \frac{1}{(1+x)^2+y^2} = \frac{s}{(1-x)^2+y^2}$$

The variable x in terms of s and y is

$$x = \frac{1+s}{1-s} \pm \sqrt{\left(\frac{1+s}{1-s}\right)^2 - y^2 - 1} \tag{4-84}$$

Substituting for x, the RMR is

$$\mathrm{RMR} = \max_{s,y} \frac{\left(1-\left(\frac{1+s}{1-s}\right) \pm \sqrt{\left(\frac{1+s}{1-s}\right)^2 - y_1^2 - 1}\right)^2 + y_1^2}{\left(1-\left(\frac{1+s}{1-s}\right) \pm \sqrt{\left(\frac{1+s}{1-s}\right)^2 - y_2^2 - 1}\right)^2 + y_2^2} \tag{4-85}$$

where

$$-.5 \le \left(\left(\frac{1+s}{1-s}\right) \pm \sqrt{\left(\frac{1+s}{1-s}\right)^2 - y^2 - 1}\right) \le .5 \qquad (4\text{-}86)$$

The variable y maximizes the equation for the RMR at y_i=0 and y_2=0.5. Substitution for y and simplification of the expression for the RMR results in the following equation

$$RMR = \max_s \frac{\left(-2s \pm .5\sqrt{(s)^2+18s-1}\right)^2 + .25(1-s)^2}{\left(-2s \pm 2\sqrt{s}\right)^2} \qquad (4\text{-}87)$$

The above equation is maximized at s=0.2 and s=5, resulting in x=0.382 and x=0.5 respectively. The RMR for both cases is 1.309. Figure 4-4 illustrates the response set used to calculate the RMR, the ray used to calculate the RMR and the set resulting when the point source response set is multiplied by the RMR.

Example 4-5. A point source of radioactivity is confined to a infinitesimally thin disk of radius R. Radiation detectors with the coordinates (r,θ,z) are located at (0,0,1) and (2,0,0) respectively. The response functions are

$$f_1 = \frac{1}{1+r^2} \text{ and } f_2 \frac{1}{4+r^2-4r\cos\theta} \qquad (4\text{-}88)$$

Using the line intersection method, the RMR for the point source response set is

$$RMR = \max_s \left(\frac{1+r_1^2}{1+r_2^2}\right), \text{ subject to: } \frac{1}{4+r^2-4r\cos\theta} = \frac{s}{1+r^2} \qquad (4\text{-}89)$$

The above constraint does not restrict the solution to the problem. The result of solving the constraint for $\cos\theta$ is

$$\cos\theta=\frac{1+r^2-4s-r^2s}{4r} \qquad (4\text{-}90)$$

For any r between zero and one, a value for $\cos\theta$ can be found that satisfies the above constraint. In addition, for $r=1$, $\cos\theta$ is indeterminate. Therefore, ignoring the constraint, the above equation for the RMR is maximized when $r_1=1$ and $r_2=0$, resulting in an RMR equal to two. When $r=0$, $(f_1,f_2)=(1,.25)$ and the slope s of the ray that determines the RMR is 0.25. Figure 4-5 illustrates the point source response set and the ray used for calculating the RMR.

Example 4-6. The source is confined to a cube centered on the origin with sides equal to one. The radioactivity is restricted to a single point source. Radiation detectors #1, #2 and #3 are located respectively at (1,0,0), (-1,0,0) and (0,1,0). The three point response functions in terms of the spatial x, y and z coordinates are

$$f_1=\frac{1}{(1-x)^2+y^2+z^2},\ f_2=\frac{1}{(1+x)^2+y^2+z^2}\ \text{and}\ f_3=\frac{1}{x^2+(1-y)^2+z^2}.$$

A given line passing through the origin is $f_1=s_1f_2$ and $f_1=s_2f_3$ or in terms of the spatial coordinates

$$\frac{1}{(1-x)^2+y^2+z^2}=\frac{s_1}{(1+x)^2+y^2+z^2}\ \text{and} \qquad (4\text{-}92)$$

$$\frac{1}{(1-x)^2+y^2+z^2}=\frac{s_2}{x^2+(1-y)^2+z^2}\ \text{where}\ -.5\le x,y,z\le .5 \qquad (4\text{-}93)$$

The above constraints define a line segment resulting from the intersection of a ray and the point source response set. This line segment can be written as two equations with y and z in terms of x

$$y=bx,\quad z^2=-x^2(1+b^2)+2x\frac{(s_1+1)}{(s_1-1)}-1 \tag{4-94}$$

$$\text{where } -.5\le x,y,z\le .5 \text{ and } b=\frac{-2s_2+s_1+1}{s_1-1}. \tag{4-95}$$

Given an s_1 and s_2, the above equation defines a curve in (x,y,z) space dependent on x. The curve, however, may not be valid in the entire domain of x since the ranges of y and z need to be satisfied. Combining the above functions with the inequalities gives the possible domain of x in terms of s_1 and s_2 as

$$x=.\frac{5}{b}y \text{ and } x=w\pm\frac{\sqrt{w^2-(b^2+1)(z^2+1)}}{b^2+1}$$
$$\text{where } w=\frac{s_1+1}{s_1-1},\ b=\frac{2s_2+s_1+1}{s_1-1},\ -.5\le y,z,\le .5. \tag{4-96}$$

The equation for the RMR is

$$\mathrm{RMR}=\max_s\max_{f,g}\frac{\sqrt{g_1^2+g_2^2+g_3^2}}{\sqrt{f_1^2+f_2^2+f_3^2}}=\max_s\max_{f,g}\left(\frac{g_n}{f_n}\right) \text{ for any } n, \tag{4-97}$$

where for g and f: $f_n=s_{N-1}f_1,\ f_{N-1}=s_{N-1}f_1,\ldots,\ f_2=s_1f_1$

Using f_1 as the selected function and substituting for y and z in terms of x, the RMR is written as

$$\mathrm{RMR}=\max_s\max_x\frac{(1-x_1)^2+b^2x_1^2-x_1^2(1+b^2)-2x_1w-1}{(1-x_2)^2+b^2x_2^2-x_2^2(1+b^2)+2x_2w-1} \tag{4-98}$$

$$=\max_s\max_s\frac{x_1}{x_2} \tag{4-99}$$

Substituting the constraints for x results in

$$\mathrm{RMR}=\max_s\max_x\left(\frac{x_2=\ \frac{5}{b}y_2 \text{ and } x_2=w\pm\frac{\sqrt{w^2-(b^2+1)(z_2^2+1)}}{b^2+1}}{x_1=.\frac{5}{b}y_1 \text{ and } x_1=w\pm\frac{\sqrt{w^2-(b^2+1)(z_1^2+1)}}{b^2+1}}\right) \tag{4-100}$$

$$\text{where } a=\frac{s_1+1}{s_1-1},\ b=\frac{2s_2+s_1+1}{s_1-1},\ -.5\le y, s, \le .5. \tag{4-101}$$

The solution to the above equation, which will not be pursued further, would require more computational effort than the other example problems that have been presented.

Techniques for Global Optimization of the RMR

The search for the RMR utilizing the above method of substitution of derivatives for the weight coefficients assists in forming a relationship between the inner and outer search; however, the problem that the gradient search method locates a local optimal solution, but not a global optimum solution, still exists. Assume that a hyperplane is locally optimized on the point source response set. If the hyperplane does not intersect the point source response set at any other location then the hyperplane is globally optimized and is a supporting plane. If however the hyperplane does intersect the point source response set then the hyperplane has not been globally optimized. The additional point or points of intersection of the nonoptimized hyperplane will assist in determining the global optimization of the hyperplane. If the hyperplane intersects the point source response set then

$$a_1f_1+a_2f_2+\dots+a_Nf_N=b \tag{4-102}$$

exists for some $(f_1,f_2,\dots,f_N)$ in the point source response set F.

Example 4-7. The radiation is confined to the line segment $-.5 \le x \le .5$ The detectors are located on the x-axis at $x=R_1$ and $x=-R_2$. The detector responses are

$$f_1=\frac{1}{(1+x)^2} \text{ and } f_2=\frac{1}{(1-x)^2} \tag{4-103}$$

The one dimensional hyperplane, or line, to be maximized is $u=2f_1+f_2$. The derivative of u with respect to x is

$$\frac{du}{dx}=-\frac{4}{(1+x)^3}+\frac{2}{(1-x)^3} \tag{4-104}$$

Above $x=0.227$, the derivative is positive. Therefore, if $x=0.3$ is the starting point for the search, u is increased by increasing x in the positive direction. At $x=0.5$, $(f_1,f_2)=(0.44,4)$ and $u=4.88$. As illustrated in Figure 4-6, the line $4.88=2f_1+f_2$ is locally optimized and intersects the point source response set at $(f_1,f_2)=(2.16,0.57)$ $(x=-0.319)$. The derivative at $x=-0.319$ is negative. Therefore, u is increased by increasing x in the negative direction. An optimum value is reached at the endpoint $x=-0.5$ where $(f_1,f_2)=(4,.444)$ and $u=8.44$.

The method relies on the ability of determining the solution of the intersection of the point source response set and an N-1 dimensional hyperplane. If the point source response set is generated from a computer code then the equation that defines the hyperplane can not be directly substituted into an equation defining the point source response set. A possible method for computer generated responses would be to search the generated point response set

for the points that satisfy the hyperplane equation with a small given error. The point that satisfies the hyperplane equation with the least error would form the new starting search point.

The disadvantage of the ray method compared to the supporting plane method for calculating the RMR of a convex set is that the ray method must search over the complete response set while the supporting plane method searches over the point source response set. In addition, the ray method is a constrained optimization while the supporting plane method has been presented as an unconstrained problem. As the following discussion illustrates, if the line intersection method is applied to a convex set, a locally optimized maximum to minimum ratio is a global optimum.

Consider a ray originating from the origin intersecting a convex set. This intersection results in a line segment of length L and a point P_{max} that is a maximum distance D_{max} from the origin and a point P_{min} that is a minimum distance D_{min} from the origin. The ratio of the maximum and minimum distances D_{max}/D_{min} is the baseline for the RMR search. If the ratio is a local maximum then the line segments in the neighborhood of the ray of the local optimum have smaller ratios than the initial ratio D_{max}/D_{min}. If it is assumed that the calculated ratio is not the global maximum ratio, then there exists another line segment with points P'_{max} and P'_{min} and a ratio D'_{max}/D'_{min} that is greater than the ratio D_{max}/D_{min}. Therefore,

the line segment connecting P'_{max} and P'_{min} is greater than the line segment connecting P_{max} and P_{min}. Since C is a convex set, the area defined by the linear combination of points P_{max}, P_{min}, P'_{max} and P'_{min} is contained within the set C. The length of line segments contained in the convex sets from rays originating from the origin increases from where the ray intersects P_{max} to where the ray intersects P'_{max}. However, it was assumed that the lines segments in the neighborhood of the line segment P_{max} to P_{min} are smaller than the above line segment. Therefore, the assumption that a global optimum exists that is different than the local optimum is incorrect.

As stated previously, the ray method searches over the convex hull instead of only the point source response set. If $f=(f_1, f_2, \ldots, f_N)$ are members of the point source response set F, the convex hull is defined as

$$C=\{c: c=\sum a_i f_i\},\ \text{where} \sum \alpha_i=1 \text{ and } \alpha \geq 1 \qquad (4\text{-}105)$$

The RMR of a convex set in terms of the ray method can be written as

$$\mathrm{RMR}=\max_d \max_{c,d} \sqrt{\frac{(c_1^2+c_2^2+\ldots+c_N^2)}{(d_1^2+d_2^2+\ldots+d_N^2)}}, \qquad (4\text{-}106)$$

where for c and d. $c_N=s_{N-1}$, $c_{N-2}=s_{N-1}c_1, \ldots, c_2=s_1 c_1$.

After substituting the constraints the above equation is written as

$$\mathrm{RMR} = \max_s \max_{c,d}\left(\frac{c_n}{d_n}\right), \text{ for any } n, \text{ where for } c \text{ and } d$$
$$c_N = s_{N-1}c_1,\ c_{N-2} = s_{N-1}c_1, \ldots, c_2 = s_1 c_1 \text{ and } c_n = \sum_i a_n f_n^i \tag{4-107}$$

where the superscript denotes different points.

A set of detector responses from a two detector point source response set is represented as (f_1, f_2). A point from the complete response set is represented as (c_1, c_2) or in terms of (f_1, f_2) as

$$c_1 = a f_1 + (1-a) f_1' \text{ and } c_2 = a f_2 + (1-a) f_2'. \tag{4-108}$$

If a given detector response point (h_1, h_2) results in the slope s, where $c_2 = sc_1$ then

$$c_2 = a f_2 + (1-a) f_2' = a s f_1 + (1-a) s f_1' \tag{4-109}$$

In terms of s and the points (f_1, f_2) and (f_1', f_2') from the point source response set, alpha is

$$a = \frac{s f_1' - f_2'}{f_2 - s f_1 + s f_1' - f_2'} \text{ where } 0 \le a \le 1. \tag{4-110}$$

Using the above equations, the RMR is written as

$$\mathrm{RMR} = \max_s \max_{a,f,g}\left(\frac{a f_1 + (1-a) f_1'}{a g_1 + (1-a) g_1'}\right) \text{ where } 0 \le a \le 1. \tag{4-111}$$

Substituting for alpha gives

$$\mathrm{RMR} = \max_s \max_{f,g}\left(\frac{\left(\frac{s f_1' - f_2'}{f_2 - f_1' - s f_2 + s f_1}\right)(f_1 - f_1') + f_1'}{\left(\frac{s g_1' - g_2'}{g_2 - g_1 - s g_2 + s g_1}\right)(g_1 - g_1') + g_1'}\right). \tag{4-112}$$

The search of the optimum detector responses can, in some cases be simplified by the substitution of the appropriate

extreme point. The fixed point will allow the entire complete response set to be evaluated by the proper choice of variables for alpha and f. If f' is the fixed point, then the search for a two detector problem over the complete response set is reduced to a single variable.

Example 4-8. The radioactivity is confined to a segment on the x axis from $x=-.5$ to $x=.5$. The detectors are located on the x axis at $x=-1$ and $x=1$ with response functions

$$f_1=\frac{1}{(1+x)^2} \text{ and } f_2=\frac{1}{(1-x)^2} \tag{4-113}$$

A point in the complete response set is written as

$$(c_1,c_2)=\left(\frac{a}{(1+x)^2}+\frac{(1-a)}{(1+x')^2},\frac{a}{(1-x)^2}+\frac{(1-a)}{(1-x')^2}\right). \tag{4-114}$$

If a given detector response results in the slope s the constraint is

$$c_2=sc_1=\frac{a}{(1-x)^2}+\frac{(1-a)}{(1-x')^2}=\frac{sa}{(1+x)^2}+\frac{s(1-a)}{(1+x')^2} \tag{4-115}$$

in terms of the two elements from the point source response set, alpha is

$$a=\frac{\frac{s}{(1+x')^2}-\frac{1}{(1-x')^2}}{\frac{1}{(1-x')^2}-\frac{s}{(1+x)^2}+\frac{s}{(1+x')^2}-\frac{1}{(1-x')}} \tag{4-116}$$

The search over two independent points from the point response set, x and x', can be reduced to a search over a single point x by recognizing that an element from the complete response set can be generated from a fixed extreme point in the point response set, such as $(f_1,f_2)=(0.444,4)$, a variable point on the point source response set and the line segment connecting

the two points. If the variable point is $(f_1,f_2)=(4,0.444)$, the line segment connecting this point with the fixed point (.444,4) creates part of the border of the complete response set.

If the $x'=.5$, then alpha is

$$a=\frac{1.778(.25s-2.25)}{\frac{-(s-1)+2x(s+1)-x^2(s-1)}{(1+x)^2(1-x)^2}+1.778(.25s-2.25)} \quad (4\text{-}117)$$

where $0\leq a\leq 1$

An alpha outside the range of (0,1) would indicate that the intersection of the line defined by the variable and fixed response points and the ray created from the given response point of unknown activity occurs outside both the line segment and the complete response set.

If the initial starting point of s is given as $s=2$, alpha is

$$a=\frac{3.112}{\frac{-1+6x-x^2}{(1+x)^2(1-x)^2}-3.112} \quad \text{where } 0\leq a\leq 1 \quad (4\text{-}118)$$

The inequality constraint limits the range of x to between $x=-.5$ where alpha=0.29, $(f_1,f_2)=(4,0.44)$ and $x=0.17$ where alpha=1 and $(f_1,f_2)=(.731,1.45)$. The equation for the RMR is written as

$$\text{RMR}=\max_a \max_{s,f,g}\left(\frac{a(f_1-.44)+.44}{a'(g_1-.44)+.44}\right) \text{ where } 0\leq a\leq 1 \quad (4\text{-}119)$$

Substituting the equations for alpha, f_1 and g_1 gives

$$\mathrm{RMR}=\max_{s}\max_{s,x}\frac{\dfrac{-3.111}{\dfrac{-1+6x-x^2}{(1+x)^2(1-x)^2}-3.111}\left(\dfrac{1}{(1+x)^2}-.44\right)+.44}{\dfrac{-3.111}{\dfrac{-1+6x-x^2}{(1+x)^2(1-x)^2}-3.111}\left(\dfrac{1}{(1+x')^2}-.44\right)+.44}$$

where $0\leq s\leq 1$.

For $s=2$ the above equation is maximized for $x=-0.5$, $(f_1,f_2)=(4,.44)$, and $x'=0.17$ $(f_1,f_2)=(0.73,1.45)$. As illustrated in Figure 4-7, the optimized detector response of the numerator is the extreme point of both the point source response set and the source container. The optimized response of the denominator has an alpha equal to one indicating that the minimized response point from the complete response set is on the point source response set. The ray intersects both the line segment that connects the two extreme points of the point response set and the point source response set itself. Possibly, the optimized ray will intersect the above line segment and the point source response set. Therefore, the numerator is kept constant at $x=-0.5$ while the denominator is allowed to vary. The optimization over the complete response set of the denominator is changed to an optimization over the point source set. The resulting equation is

$$\mathrm{RMR}=\max_{s,x}\left(\frac{\dfrac{.44-4s}{-3.56-3.56s}(3.56)+.44}{\dfrac{1}{(1+x)^2}}\right)\quad \text{where } 0\leq s\leq 1 \tag{4-121}$$

The intersection of the point source response set with a ray defined by $f_2=sf_1$, determines the following x,

$$s=\frac{f_2}{f_1}=\frac{(1+x)^2}{(1-x)^2} \text{ or } (s-1)x^2-2(s+1)x+(s-1)=0$$

$$\text{and } x=\frac{(s+1)\pm 2\sqrt{s}}{(s-1)}=\frac{\sqrt{s}\mp 1}{\sqrt{s}\pm 1}. \qquad (4\text{-}122)$$

The RMR is written as

$$\mathrm{RMR}=\max_s \frac{\left(\frac{4.44s}{1+s}\right)}{\left(1-\frac{\sqrt{s}\pm 1}{\sqrt{s}\mp 1}\right)^2}=\max_s\left(\frac{17.76s}{2s\pm 2\sqrt{s}+1+s^2\pm 2(\sqrt{s})^3}\right). \qquad (4\text{-}123)$$

After setting the derivative of the RMR equal to zero and collecting terms, the variable s will optimize the RMR for

$$1-s^2\mp s^{1.5}\pm s^{.5}=(\pm s^{.5}+1)(\mp s^{1.5}+1)=0 \qquad (4\text{-}124)$$

or at $s=1$ (RMR=2.22).

Example 4-9. The radioactivity is confined to an infinitesimally thin disk of radius R. Two detector readings are performed above the centerline of this disk.

Using cylindrical coordinates, the equation of the disk that confines the activity is $r=R$. The detector readings are performed at $z=Z_1$ and $z=Z_2$ with $Z_2>Z_1$. The response functions are

$$f_1=\frac{1}{(Z_1^2+r^2)} \text{ and } f_2=\frac{1}{(Z_2^2+r^2)} \qquad (4\text{-}125)$$

A point in the complete response set in written as

$$(c_1,c_2)=\left(\frac{\alpha}{Z_1^2+r^2}+\frac{(1-\alpha)}{Z_1^2+(r')^2},\frac{\alpha}{Z_2^2+r^2}+\frac{(1-\alpha)}{Z_2^2+(r')^2}\right). \qquad (4\text{-}126)$$

Example 4-8 solved the RMR analytically by noting the behavior of x and alpha. In this example, curvature of the point source response set will assist in determining the RMR. The second derivative of f_2 with respect to f_1 is

$$\frac{d^2f_2}{df_1^2}=\frac{-2}{\left(f_1+(Z_2^2-Z_1^2)\right)^2}<0. \tag{4-127}$$

Therefore, the point response function of f_2 in terms of f_1 is concave downwards and the upper boundary of the complete response set is the point response set and the lower boundary of the complete response set is the line segment connecting the endpoints of the point response set. The boundaries of the complete response set are the point response set and the line segment

$$(c_1,c_2)=(\alpha f_1(0)+(1-\alpha)f_2(1),\alpha f_2(0)+(1-\alpha)f_2(1))$$
$$=\left(\frac{\alpha}{Z_1^2}+\frac{(1-\alpha)}{Z_1^2+1},\frac{\alpha}{Z_2^2}+\frac{(1-\alpha)}{Z_2^2+1}\right)\text{ where }0\le\alpha\le1. \tag{4-128}$$

The equation of the line containing this segment is

$$f_2=wf_1+b$$
$$\text{where } w=\frac{Z_1^2(Z_1^2+1)}{Z_2^2(Z_2^2+1)}\text{ and } b=\frac{Z_2^2-Z_1^2}{Z_2^2(Z_2^2+1)}. \tag{4-129}$$

The ray from the origin to the point (h_1,h_2) intersects the line segment at

$$f_2=\frac{bh_1}{h_2-wh_1},\ f_1=\frac{bh_2}{h_2-wh_1} \tag{4-130}$$

Therefore, f_1 and f_2, in terms of Z_1 and Z_2, are

$$f_1=\frac{h_1(Z_1^2-Z_2^2)}{h_2Z_2^2(Z_2^2+1)-h_1Z_1^2(Z_1^2+1)},\ f_2=\frac{h_2(Z_1^2-Z_2^2)}{h_2Z_2^2(Z_2^2+1)-h_1Z_1^2(Z_1^2+1)}$$

A ray from the origin to the point (h_1,h_2) intersects the point source response set at

$$f_1=\frac{h_1-h_2}{h_2(Z_2^2-Z_1^2)},\ f_2=\frac{h_1-h_2}{h_1(Z_2^2-Z_1^2)}. \tag{4-132}$$

A ray passing through the set C connecting the origin and the

point (h_1, h_2) intersects the boundaries of the set C on the point source response set and the line segment connecting the extremes of the point source response set. The ratio of the distances from the origin of the intersection of the line segment with the intersection of the point source response set is

$$ratio = \frac{\dfrac{h_1 - h_2}{h_2 (Z_2^2 - Z_1^2)}}{\dfrac{h_1 (Z_1^2 - Z_2^2)}{h_2 Z_2^2 (Z_2^2 + 1) - h_1 Z_1^2 (Z_1^2 + 1)}} \tag{4-133}$$

which simplifies to

$$ratio = \frac{(h_1 - h_2)\left(h_2 Z_2^2 (Z_2^2 + 1) - h_1 Z_1^2 (Z_1^2 + 1)\right)}{h_1 h_2 \left(Z_1^2 - Z_2^2\right)^2} \tag{4-134}$$

Multiplying the above equation by $h_1 h_2 / h_1 h_2$ results in

$$ratio = \frac{\left(1 - \dfrac{h_2}{h_1}\right)\left(Z_2^2 (Z_2^2 + 1) - \dfrac{h_1}{h_2} Z_1^2 (Z_1^2 + 1)\right)}{\left(Z_1^2 - Z_2^2\right)^2} \tag{4-135}$$

which is written in terms of the slope s as

$$ratio = \frac{(1 - s)\left(s Z_2^2 (Z_2^2 + 1) - Z_1^2 (Z_1^2 + 1)\right)}{s\left(Z_1^2 - Z_2^2\right)^2} \tag{4-136}$$

The RMR is the maximized ratio over s of the above equation and is written as

$$\mathrm{RMR} = \max_s \frac{(1 - s)\left(s Z_2^2 (Z_2^2 + 1) - Z_1^2 (Z_1^2 + 1)\right)}{s\left(Z_1^2 - Z_2^2\right)^2}. \tag{4-137}$$

If Z_1=1 and Z_2=1.414, the RMR is

$$\mathrm{RMR} = \max_s \left(-6s + 8 - \frac{2}{s}\right). \tag{4-138}$$

Setting the derivative equal to zero gives

$$\frac{d(\mathrm{RMR})}{da} = -6 + \frac{2}{a^2} = 0. \qquad (4\text{-}139)$$

The RMR is determined from a=0.577 (RMR=1.072). The point source response set and the ray used to calculate the RMR are illustrated in Figure 4-8. Since the RMR was searched by the ray method, the solution found is the RMR and not only a local optimized ratio.

The Utilization of the RMR for the Optimization of Detector Systems

As mentioned in the introduction, the RMR can aid in assessing the performance of a detector system. Such a performance criterion can be useful for both the operator and designer of detector systems. While the calculational methodology presented in this chapter is sufficient for assessing the RMR of a fixed position detector system, the more complex problem of the design a detector system and the proper detector placement of a variable position detector system is not addressed in depth in this dissertation. The technique of detector system design should include the resulting information that is available after the RMR has been calculated in order to attempt and improvement in the RMR. If the line intersection method is used, the slopes of the ray should be utilized to compare the relative strengths of the detectors that contribute to the RMR. If a is formulated as $f_2 = af_1$, then $a = f_2/f_1 = \tan\theta$. The normalized square of the magnitudes of the vectors f_1 and f_2 that contribute to the ray

defined by the slope s are

$$\mathrm{norm}(f_1^2) = \frac{f_1^2}{f_1^2+f_2^2} = \cos^2\theta = \cos^2(\tan^{-1}s) \text{ and } \mathrm{norm}(f_2) = \sin^2(\tan^{-1}s)$$

The normalised square of the magnitudes give the relative strengths of the detectors that contribute to the RMR. For example, the previous example gave an s of 0.577, which results in $\mathrm{norm}(f_1^2)=0.75$ and $\mathrm{norm}(f_2^2)=0.25$. The relative contributions of the detectors to the RMR are 75% for f_1 and 25% for f_2. An improved measurement configuration could possibly have detector #2 located to a position that is less sensitive to positional variations of the source positions.

Very close to the source container the spatial uncertainty of the activity contributes significantly to the measurement uncertainty. As the measurement location increases in distance from the source container, the statistical uncertainty of the response increase resulting in the optimal location of a detector measurement to be located between a measurement flush with the container and a measurement an infinite distance from the container. For simple detector/source container cases, functional expressions can describe the relationship between the RMR and the location of the measurement. Appendix B contains a development of an optimization for a one and a two detector configuration utilizing a line source container.

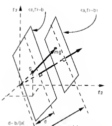

Figure 4-1. Respective supporting planes for two responses that are the result of identical distributions and different activities.

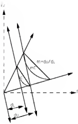

Figure 4-2. Arbitrary supporting planes for a convex set.

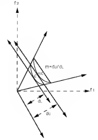

Figure 4-3. Proper supporting planes for determining the RMR of a convex set.

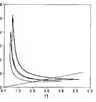

Figure 4-4. Utilization of the line intersection method for a non convex set.

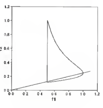

Figure 4-5. Line intersection method for complete response set in Figure 4-2.

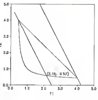

Figure 4-6. Method of searching for the global optimum utilizing supporting planes.

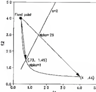

Figure 4-7. Utilization of a fixed point search.

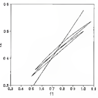

Figure 4-8. Utilization of the line intersection method for a convex set.

CHAPTER 5

ASYMPTOTIC SOLUTIONS

Increasing number of detectors to infinity does not guarantee that the assay system will be able to distinguish the amount of activity perfectly (RMR=1). As the number of symmetric detector increases, the RMR can converge to some limiting value greater than one.

Example 5-1. Consider line source container located on the x axis between -0.5 and 0.5. A number of symmetrically placed detectors are located on the midplane of the source container a distance R from the origin with a response

$$f_x = \frac{1}{(R^2+x^2)}. \tag{5-1}$$

If the number of symmetric detectors is increased to infinity, the detector system cannot distinguish between a unit point source located at the origin and a point source of strength $(R^2+.25)/R^2$ located at $x=.5$. If the source container of the above example is changed to a 55 gallon drum, increasing the number of midplane symmetric detectors would aid in distinguishing source positions perpendicular to the lateral axis but not in distinguishing point source distributions

along the axis. Increasing the number of detectors to infinity in this case would cause the RMR to approach the asymptotic value of $(R^2+.25)/R^2$.

The above asymptotic value of the RMR was obtained by inspection of the source container and the detector response equations. The asymptotic value can be approximated by evaluating the RMR as the number of detectors becomes increasingly large. For a generalized detector system, a technique for the direct evaluation of the asymptotic RMR has not been formulated although the usefulness of such a technique has been mentioned (Ben-Haim 1985b).

This chapter develops a method for the evaluation of the asymptotic value of the RMR as the number of detectors increase to infinity. The analytical solution for the asymptotic RMR is solved by converting the set in detectors that increase to infinity to an integral. The following method is for a set of symmetrical detector increasing to infinity along with a set of detectors which are not part of the symmetric design. Using the supporting plane method, the RMR for the N-1 symmetrically placed detectors along with an additional nonsymmetrically placed detector is written as

$$\mathrm{RMR} = \min_a \max_{f,g,f',g'} \left(\frac{a_1 f_1 + a_2 f_2 + \cdots + a_{N-1} f_{N-1} + a_N f_N}{a_1 g_1 + a_2 g_2 + \cdots + a_{N-1} f_{N-1} + a_N f_N} \right) \qquad (5\text{-}2)$$

Since the first N-1 detectors are symmetric, the corresponding weight coefficients are equal. Taking into account that

$a_1=a_2=\ldots a_{N-1}$ the equation for the RMR is

$$RMR=\min_a \max_{f,g,f',g'}\left(\frac{a_1\sum_{n=1}^{n=N-1}(f_n)+a_N f_N}{a_1\sum_{n=1}^{n=N-1}(g_n)+a_N f_N}\right) \tag{5-3}$$

In order to convert the summations to integrals, the numerator and denominator are multiplied by Δn resulting in

$$RMR=\min_a \max_{f,g,f',g'}\left(\frac{a_1\sum_{n=1}^{n=N-1}(f_n\Delta n)+a_N f_N\Delta n}{a_1\sum_{n=1}^{n=N-1}(g_n\Delta n)+a_N f_N\Delta n}\right) \tag{5-4}$$

The limit of the first term of the numerator and denominator as $N\to\infty$ and $\Delta n\to 0$ is the integral f_s and g_s respectively. The value of the second terms, $a_N f_N\Delta n$ and $a_N g_N\Delta n$ is not necessarily zero as $\Delta n\to 0$ since the coefficient a_N can increase as much as necessary. The term $a_N f_N\Delta n$ can be changed to a summation by allowing the number of detectors at the location of the f_N detector to increase to infinity. With this additional summation, the RMR is written as

$$RMR=\min_a \max_{f,g,f',g'}\left(\frac{a_1\sum_{n=1}^{n=N-1}(f_n\Delta n)+a_N\sum f_N\Delta n}{a_1\sum_{n=1}^{n=N-1}(g_n\Delta n)+a_N\sum f_N\Delta n}\right) \tag{5-5}$$

The limit of the above equation as $N\to\infty$ and $\Delta n\to 0$ is

$$\mathrm{RMR} = \min_{s} \max_{f,g,f',g'} \left(\frac{a_1 \int_{n=1}^{\infty} f_n \, dn + a_N \int f_N \, dn}{a_1 \int_{n=1}^{\infty} g_n \, dn + a_N \int f_N \, dn} \right) \tag{5-6}$$

As demonstrated in the following example, the integral can be solved analytically by converting the variable n to the spatial parameters of the problem.

Example 5-2. Consider a source confined to a circle of radius one centered on the origin. The N detectors are located symmetrically on a circle of radius two. Assume that the Nth detector is fixed at (2,0). Figure (5-1) illustrates the case of four symmetrically placed detectors surrounding a disk. The source location that corresponds to the minimum detector response for any number of detectors greater than one is located at the center of the disk (0,0). The maximized detector response for the nth detector located at $(r,\theta) = (1, \frac{2\pi n}{N})$ is

$$f_n = \frac{1}{5-4\cos\theta} \tag{5-7}$$

The asymptotic limit for the RMR in terms of the spatial parameter is

$$RMR = \frac{\frac{2\pi}{N}\int_0^{2\pi}\frac{1}{5-4\cos\theta}d\theta}{\frac{2\pi}{N}\int_0^{2\pi}\frac{1}{4}d\theta} \qquad (5\text{-}8)$$

The solution to the above equation is 4/3. A numeric calculation of eighteen symmetrically placed detectors results in an RMR of 1.337.

The above solution considers the multiple responses as multiple detectors. Other perspectives can be taken which will result in the same formulation of the problem. For time invariant problems, a single detector at multiple positions can be considered identical to multiple detectors at fixed locations. Another alternative would be to have a single detector remain stationary and have the source container move to another position or rotate about an axis. Such a rotation has been previously considered (Martens and Fiies 1990) as a means of reducing the uncertainty assessment.

As the number of the symmetric detectors approach infinity the resulting "infinite-symmetric" detector is represented by an integral which is equivalent to a single detector with the same geometry that the infinite number of point detectors form. In the above example, the infinite number of detectors surrounding the disk is equivalent to a ring detector. In the case of the above example, the problem can be redefined as a rotating source container and a detector located somewhere on the circle r=2. The response function for

the above circular source container rotating at the origin with a point source located at (r,θ) is

$$f_r=\int_0^{2\pi}\frac{1}{R^2+r^2-Rr\cos\theta}d\theta. \tag{5-9}$$

Multiplying and dividing by R^2 gives

$$f_r=\frac{1}{R^2}\int_0^{2\pi}\frac{1}{1+\frac{r^2}{R^2}-\frac{r}{R}\cos\theta}d\theta. \tag{5-10}$$

A table of definite integrals (Spiegel 1968, 15.47) provides the following solution

$$f_r=\frac{2\pi}{R^2-r^2}. \tag{5-11}$$

The minimum detector response occurs at r=0 and the maximum response occurs at the edge of the source container giving an RMR of

$$\mathrm{RMR}=\frac{R^2-0}{R^2-(r_{max})^2} \tag{5-12}$$

which for a circular source container of radius one and the detector placed at R=2 results in an RMR of 4/3.

The asymptotic limit of the RMR can assist the assessor in developing a measurement strategy that maximizes the efficiency of the detector measurements while minimizing the time and effort necessary to achieve those measurement. The RMR for an infinite number of detector in the above example was 4/3. This asymptotic limit may be achieved by a rotation

of the container or a ring detector. Four symmetrical measurements around the disk achieve an RMR of 1.51 and eight symmetric detector achieve an RMR of 1.35. Therefore, the assessor can determine whether the time and expense of increasing the number of detector measurements is worthwhile.

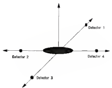

Figure 5-1. Four symmetric detectors surrounding a disk source container.

CHAPTER 6
CONCLUSION

The application of Euclidean space to the problem of assessment of radioactive material in the presence of spatial uncertainty has achieved limited use as both a research and an industry tool. More common is the use of methods such as segmentation or rotation of the container and a linear combination of responses. These methods, while able to reduce the uncertainty, do not have the flexibility to adjust to modifications in container geometry, detector positions or number of detectors. As a corollary, the above methods do not allow for an expansion to concepts of optimization and feedback control.

Without application of Euclidean space, the equations for interpretation and optimization rely on factors such as the symmetry the detector source configuration, assumed spatial distributions of the source and a loss of information (through averaging or a linear combination of the multiple responses) to allow for a solution to the problem. The application of dimensional space allows for a mathematical framework independent of container geometry, measurement position and source distribution. This dissertation has expanded the

application of Euclidean space to develop solutions for problems that have previously been unsolved, such as an analytical solution to a asymptotic limit, the utilization of nonconvex response sets, the global optimization of the RMR and the interpretation of multiple measurements. The results are taken a step further by combining the interpretation equation with the RMR equation to develop a feedback optimization procedure where the results of the previous measurements aid in determining the position of a future measurement.

A possible reason for the limited use of a vector space approach for multiple measurements is the lack of tools necessary for the assessment of the radioactive material measured. The formulations developed in this dissertation, especially the interpretational method, demonstrate that the application of a multidimensional space provides generalized solutions to a variety of assessment problems.

Since a discrepancy exists between the potential of the application of vector space and the number of references available on the subject, there are other significant accomplishments incorporated in the dissertation. The inclusion of simplified examples, the "filling in" of information assumed in other literature, the "walking through" of results, the emphasis on unifying concepts and interpreting problems from multiple perspectives should allow

this dissertation to serve as a reference and tutorial for would-be practitioners and researchers.

APPENDIX A
EQUATIONS FROM ANALYTICAL GEOMETRY AND SET THEORY

A vector in N-dimensional space originating from the origin to a point is written as $(f_1, f_2, \ldots, f_N)$ or f. The magnitude of a vector f is written as $|f|$. A direct result of the parallel postulate in Euclidean space is that the distance d between two points $(f_1, f_2, \ldots, f_N)$ and $(g_1, g_2, \ldots, g_N)$ is

$$d=\sqrt{(g_1-f_1)^2+(g_1-f_2)^2+\ldots+(g_N-f_N)^2} \text{ (Faber 1983).} \quad \text{(A-1)}$$

Directional cosines of a line joining points f and g are

$$l=\cos\beta_1=\frac{g_1-f_1}{d},\ m=\cos\beta_1=\frac{g_2-f_2}{d},\ldots,n=\cos\beta_N=\frac{g_N-f_N}{d} \quad \text{(A-2)}$$

Figure A-1 illustrates the directional cosines for a line joining points f and g.

An N-1 dimensional hyperplane in N dimensional space, is defined by the vector $a=(a_1, a_2, \ldots, a_N)$ and a point $(f_1, f_2, \ldots, f_N)$. The equation of an N dimensional plane is

$$a_1f_1+a_2f_2+\ldots+a_Nf_N=\langle a, f\rangle=b. \quad \text{(A-3)}$$

The normal form of the equation for an N-1 dimensional hyperplane is

$$\cos\beta_1f_1+\cos\beta_2f_2+\ldots+\cos\beta_Nf_N=p. \quad \text{(A-4)}$$

The perpendicular distance from the origin to the plane at point P is p. The angles $\beta_1, \beta_2, \ldots, $and β_N are defined by the ray from the origin to the point P and the respective axes defining the N dimensional space. A example of a plane intersected by a line from the origin is show in figure A-2. One method of evaluating the perpendicular distance from a plane $b=\langle a,f\rangle$ to the origin is found by calculating the magnitude of the vector f that is a minimum. Since

$$b=\langle a,f\rangle=|a||f|\cos\theta \tag{A-5}$$

the minimum of f is

$$\min|f|=\min\frac{b}{|a|\cos\theta}=\frac{b}{|a|}=\frac{\langle a,f\rangle}{|a|}, \tag{A-6}$$

Since the equation of the line that is perpendicular to the plane $\langle a,f\rangle=b$ is

$$\frac{f_1}{a_1}=\frac{f_2}{a_2}=\ldots=\frac{f_N}{a_N} \text{ (Spiegel 1968)}, \tag{A-7}$$

the perpendicular distance from the origin to the plane can be determined by calculating the point of intersection of the above line with the plane $\langle a,f\rangle=b$.

A group a points or vectors is designated as a set. A set is consider compact if any curve connecting points from the set continuous. A line segment joining any two points $(g_1, g_2, \ldots, g_N)$ and $(b_1, b_2, \ldots, b_N)$ is defined as a collection of points $(f_1, f_2, \ldots, f_N)$ such that

$$f=(ag+(1-a)h) \text{ where } 0 \le a \le 1 \qquad \text{(A-8)}$$

A set F is convex if for each pair of points $(g_1, g_2, \ldots, g_N)$ and $(h_1, h_2, \ldots, h_N)$ in the set C, the line segment connecting these points is also in the set C. A convex combination of M points, $g, h, \ldots, k$, is defined as the set of points $(c_1, c_2, \ldots, c_N)$ such that

$$c=a_1g+a_2h+\ldots+a_Mk \text{ where } \sum_M a_n=1 \text{ and all } a_n \ge 1. \qquad \text{(A-9)}$$

If the number of points M approaches infinity, the constraint of the coefficient is

$$\int a_n dn=1 \text{ and all } a \ge 1. \qquad \text{(A-10)}$$

The set of points $(c_1, c_2, \ldots, c_N)$ of the set C which are the convex combination of points of a set F composed of points $(f_1, f_2, \ldots, f_N)$ is represented as

$$C=\{c.c=\int a_n f dn \text{ such that } \int a_n dn=1 \text{ and } a_n \ge 1\} \qquad \text{(A-11)}$$

The relationship between a convex set and a convex combination of points is as follows: a set is convex if and only if every convex combination of points in S lies in S (Sen-Haim 1985a).

A convex hull of a set S is defined as the intersection of all the convex sets which contain S and is denoted as the convex hull of S. A convex hull is the smallest convex set that contains S. In addition, S is convex if and only if S equals the convex hull of S (Lay 1982). The convex hull of S

consists of exactly all convex combinations of elements in the set S. A supporting plane of a set S is defined as the plane that has at least one point in common with the set S and is a maximum or minimum distance from the origin. The upper supporting plane is a maximum distance from the origin and the lower supporting plane is a minimum distance from the origin. A supporting plane of the set S is written as

$$\max(\text{or } \min)_{b \in S} \langle a, b \rangle. \qquad \text{(A-12)}$$

A supporting plane in 2 dimension space is a line. Figure A-3 illustrates an example of an upper and lower supporting plane for a two dimensional convex set.

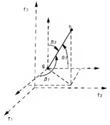

Figure A-1. Directional cosines for a line.

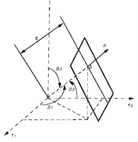

Figure A-2. Plane intersected by a line from the origin.

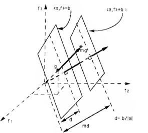

Figure A-3. Upper and lower supporting planes for a convex set.

APPENDIX B
ADAPTIVE ASSAY WITH STATISTICAL AND SPATIAL UNCERTAINTY

The dissertation has presented methods for the interpretation of detector responses and the calculation of the RMR. Statistical uncertainty of the detector responses has so far been ignored, although the formulations for the RMR that include statistical uncertainty have been derived (Ben-Haim 1985b, pg 65).

Any unrestricted detector optimization will be required to take into account the statistical uncertainty of the detector responses. Without the incorporation of statistical uncertainty the ideal detector configuration will frequently be the case of the detectors located an infinite distance from the source container. The RMR, with the inclusion of statistical uncertainty, is written as $\underline{\mathrm{RMR}}$. The $\underline{\mathrm{RMR}}$ is dependent not only on the positions of the detectors, but the amount of source material that is present in the sample container. If the RMR is

$$\mathrm{RMR} = \frac{\langle h, v \rangle}{\langle h, u \rangle} \qquad \text{(B-1)}$$

the quantities $m+\delta$ and m are distinguishable for a measurement of duration t, if and only if

$$\langle h, (m+\delta)tu\rangle > \langle h, mtv\rangle \tag{B-2}$$

If σ is the statistical standard deviation, then the RMR is based on the equation

$$\langle h, (m+\delta)tu\rangle - b\sigma(\langle h, (m+\delta)tu\rangle) > \langle h, mtv\rangle + b\sigma(\langle h, mtv\rangle) \tag{B-3}$$

where b is a positive number indicating the level of statistical confidence required. Solving the above equation for the RMR results in

$$\mathrm{RMR} = \frac{m+\delta}{m} = \left(\sqrt{\mathrm{RMR}+V^2+T}+V\right)^2 \tag{B-4}$$

where

$$T = \frac{b\sigma(\langle h, v\rangle)}{\langle h, u\rangle\sqrt{mt}} \text{ and } V = \frac{b\sigma(\langle h, v\rangle)}{2\langle h, u\rangle\sqrt{mt}} \tag{B-5}$$

Note that the amount of source material present, which is to be interprated from the detector measurements, is required for the determination of the RMR. If the amount of mass can be approximated, the detectors can be optimized prior to measuring the sample. Since the amount of radioactive material is the variable that is being assessed, however, the mass may not be known with any degree of accuracy. The assessment of the amount of radioactivity present from an initial detector configuration provides information in order that the locations of the detectors can be changed. The new locations of the detectors will result in a lower RMR for the detector system and possibly a reduced range of the interpreted amount of radioactive material present. The initial measurements can be

incorporated into the readjusted measurements by forming an $N+R$ detector space where N is the number of initial measurements and R is the number of relocated detector measurements.

A brief remark was previously presented (Ben-Haim 1985b, pg. 305) where the design method of minimizing the RMR is performed by adjusting the Nth detector position based on data from the previous N-1 detector positions. This suggestion utilizes the previous N-1 detector measurements taken, incorporates these measurement within an N detector space and attempts an optimization of the Nth detector response. For example, a single detector interprets the amount of radioactive material from the detector response. The assessment of mass indicates that the detector position should be changed. This new detector location provides an RMR that is lower than the RMR from the previous detector location. However, if the source configuration is time independent, the previous detector response and the optimized detector response can be considered as a two detector set-up. The RMR for this two detector set-up will be lower or equal to the optimized single detector set-up.

The following example problem confines the radioactivity to a line of unit length along the x-axis. The optimal location of the detector is found by utilizing the information from the first measurement and creating a two dimensional response space. The detector response located a distance R_1 to

the left of the origin is

$$f_1 = \frac{1}{(R_1+x)^2} . \qquad \text{(B-6)}$$

The minimum detector response occurs for a source location at x=0.5. the maximum detector response occurs for the source located at x=-0.5; therefore, the RMR is

$$\text{RMR} = \frac{f_{max}}{f_{min}} = \left(\frac{R_1+.5}{R_1-.5}\right)^2 \qquad \text{(B-7)}$$

If counting statistics are included with b and t assumed to be 1, the equation for the $\underline{\text{RMR}}$ is

$$\underline{\text{RMR}} = \left(\sqrt{\text{RMR}} + \frac{1}{\sqrt{m}}\right)^2 = \left(\frac{R_1+.5}{R_1-.5} + \frac{R_1+.5}{\sqrt{m}}\right)^2 . \qquad \text{(B-8)}$$

Setting the derivative of the $\underline{\text{RMR}}$ to zero gives

$$\frac{d\underline{\text{RMR}}}{dR_1} = 2\left(\frac{R_1+.5}{R_1-.5} + \frac{R_1+.5}{\sqrt{m}}\right)\left(\frac{-1}{(R_1-.5)^2} + \frac{1}{\sqrt{m}}\right) \qquad \text{(B-9)}$$

The derivative is zero for

$$\frac{1}{(R_1-.5)^2} = \frac{1}{\sqrt{m}} \text{ or } R_1 = m^{.25} + .5. \qquad \text{(B-10)}$$

The optimal detector position is dependent on the amount of activity present. Without any prior information available, m is assumed to be one and the detector is positioned at x=-1.5. The RMR with and without statistical uncertainty is 4 and 9 respectively. The detector response is 0.889. The mass is predicted to be between 0.889/1=0.889 and 0.889/0.25=3.556.

The average of the two responses is 2.223. Using the above equation, a single detector should be located at R_i=1.721.

In order to incorporate the previous measurement of 0.889 of a future measurement, a two detector system is established with the first detector fixed at x=1.5. The optimum location of the second detector may not be at R_i=1.721. Assume that the readjusted detector is located somewhere on the positive x-axis. The detector response is

$$f_2 = \frac{1}{(R_2 - x)^2} \qquad \text{(B-11)}$$

The function will be attempted to be classified as concave or convex by solving for the second derivative. The response of detector 1 in terms of detector 2 is

$$f_2 = \frac{1}{\left(R_2 - \left(\frac{1}{\sqrt{f_1}} - 1.5\right)\right)^2} = \frac{f_1}{\left(\sqrt{f_1}(R_2 + 1.5) - 1\right)^2} \qquad \text{(B-12)}$$

The derivative is

$$\frac{df_2}{df_1} = \frac{\left(\sqrt{f_1}(R_2+1.5)-1\right)^2 - 2f_1\left(\sqrt{f_1}(R_2+1.5)-1\right)\left(\frac{1}{2\sqrt{f_1}}(R_2+1.5)\right)}{\left(\sqrt{f_1}(R_2+1.5)-1\right)^2}$$

Simplifying the expression gives

$$\frac{df_2}{df_1} = \frac{\left(\sqrt{f_1}(R_2+1.5)-1\right)^2 - \left(\sqrt{f_1}(R_2+1.5)-1\right)\left(\sqrt{f_1}(R_2+1.5)\right)}{\left(\sqrt{f_1}(R_2+1.5)-1\right)^2} \qquad \text{(B-14)}$$

and

$$\frac{df_2}{df_1} = \frac{-3\left(\sqrt{f_1}(R_2+1.5)-1\right)}{\left(\sqrt{f_1}(R_2+1.5)-1\right)^2} \qquad \text{(B-15)}$$

The numerator of the second derivative is

$$num\frac{d^2f_2}{df_1^2} = \left(\frac{3\,(R_2+1.5)}{2\sqrt{f_1}}\right)\left(\sqrt{f_1}(R_2+1.5)-1\right)+\left(\sqrt{f_1}(R_2+1.5)-1\right)^2\left(\frac{3\,(R_2+1.5)}{\sqrt{f_1}}\right)$$

Factoring out gives

$$num\frac{d^2f_2}{df_1^2} = -\left(\frac{3\,(R_2+1.5)}{\sqrt{f_1}}\right)*\left[\left(.5\sqrt{f_1}(R_2+1.5)-1\right)^2-\left(\sqrt{f_1}(R_2+1.5)-1\right)^2\right]$$

and

$$num\frac{d^2f_2}{df_1^2} = \left(-\frac{3\,(R_2+1.5)}{\sqrt{f_1}}\right)*\left[\left(.5\sqrt{f_1}(R_2+1.5)-1\right)^2\right] \qquad \text{(B-18)}$$

which is greater than zero; therefore, the function is convex and the upper supporting plane will intersect the endpoints and the lower supporting plane will be tangent to the point source response function.

The endpoints of the point source response function occur at $x=-.5$ and $x=.5$. The slope of the line connecting these points is

$$s = \frac{f_2(.5)-f_2(-.5)}{f_1(.5)-f_1(-.5)} = \left(\frac{8}{3}\right)\frac{-R_2}{(R_2-.5)^2(R_2+.5)^2}. \qquad \text{(B-19)}$$

The derivative of f_2 with respect to f_1 in terms of x is

$$\frac{df_2}{df_1} = \frac{df_2}{dx}\,\frac{dx}{df_1} = \frac{2}{(R_2-x)^2}\,\frac{(1.5+x)^3}{1} = \frac{(1.5+x)^3}{(R_2-x)^2} \qquad \text{(B-20)}$$

The line of slope s is tangent at $x=x_s$ where x_s is found from the equation

$$\frac{(1.5+x_s)^3}{(R_2-x_s)^3}=\left(\frac{8}{3}\right)\frac{-R_2}{(R_2-.5)^2(R_2+5)^2}. \tag{8-21}$$

Solving for x_s gives

$$x_s=\frac{R_2\left(\sqrt[3]{\frac{8R_2}{3(R+.5)^2(R_2-.5)^2}}\right)-1.5}{1+\sqrt[3]{\frac{8R_2}{3(R_2+.5)^2(R_2-.5)^2}}} \tag{8-22}$$

The RMR is

$$\mathrm{RMR}=\frac{sf_1(.5)-f_2(-.5)}{-sf_1(x_s)+f_2(x_s)}=\frac{\frac{8R_2}{3(R_2-.5)^2(R_2+.5)^2}+\left(\frac{1}{R_2+.5}\right)^2}{\frac{8R_2}{3(R_2-.5)^2(R_2+.5)^2}\left(\frac{1}{1.5+x_s}\right)^2+\left(\frac{1}{R_2-x_s}\right)^2}$$

The RMR with statistical uncertainty is

$$\mathrm{RMR}=\left(\sqrt{\mathrm{RMR}+V^2+T}+V\right)^2 \tag{8-24}$$

where

$$T=\frac{\sqrt{\left(\frac{8R_2}{3(R_2-.5)^2(R_2+.5)^2}\right)^2+\left(\frac{1}{R_2+.5}\right)^2}}{\frac{8R_2\sqrt{m}}{3(R_2-.5)^2(R_2+.5)^2(1.5+x_s)^2}+\frac{\sqrt{m}}{(R_2-x_s)^2}} \tag{8-25}$$

and

$$V=\frac{\sqrt{\left(\frac{8R_2}{3(R_2-.5)^2(R_2+.5)^2}\right)^2\left(\frac{1}{1.5+X_s}\right)^2+\left(\frac{1}{R_2-X_s}\right)^2}}{\frac{16R_2\sqrt{m}}{3(R_2-.5)^2(R_2+.5)^2(1.5+X_s)^2}+2\frac{\sqrt{m}}{(R_2-X_s)^2}}. \qquad \text{(B-26)}$$

The above equation is dependent only on R_2 and m. Using one detector, the amount of activity present was calculated to be between 0.889 and 3.556, an average of 2.223. The optimal location for the detector is found by substituting an estimated value for m, setting the derivative of the RMR with respect to R_2 equal to zero and solving for R_2.

Since the previous detector response of f_1=0.889 will be used for the interpretation of the amount of radioactivity present, the above solution determined the RMR without regard to the information from the previous detector response. The response of detector #1 limits the assessment uncertainty present. If the ray method is utilized for determining the RMR of the above example, the ray intersecting the response set is limited to $(0.889, f_2)$ where f_2 is confined to the complete response set multiplied by the estimated amount of radioactivity. The restriction, f_1=0.889, places the point of intersection of the ray at $(0.889, f_2)$ where f_2 could result in the point being on the boundary of the complete response set or a point within the complete response set. The equation of the point source response of detector #2 is dependent only on the location of the detector and the source amount present

$$f_2 = \frac{f_1 m}{\left(\sqrt{f_1}(R_2+1.5)-\sqrt{m}\right)^2} \tag{B-27}$$

For purposes of calculating the RMR the value of m can be considered one since the RMR of the identical regardless of the amount of radioactive material assumed. Assuming f_1=0.889 and m=1, the equation of the ray that intersects the point is

$$f_2 = \frac{f_1}{.943R_2+.414} \tag{B-28}$$

The equation of the line that intersects the two endpoints of the point source response set is

$$f_2 = \frac{-8f_1R_2}{3(R_2-.5)^2(R_2+.5)^2} + \left(\frac{1}{R_2+.5}\right)^2 + \left(\frac{8R_2}{3(R-.5)^2(R_2+.5)^2}\right) \tag{B-29}$$

The intersection of this line, which bounds the complete response set, with the ray is

$$f_1 = \frac{\left(\frac{1}{R_2+.5}\right)^2 + \left(\frac{8R_2}{3(R-.5)^2(R_2+.5)^2}\right)}{\left(\frac{1}{.943R_2+.414} + \frac{8R_2}{3(R_2-.5)^2(R_2+.5)^2}\right)} \tag{B-30}$$

The intersection with the point source response set occurs at

$$.943R_2+.414 = \left(\sqrt{f_1}(R_2+1.5)-1\right)^2. \tag{B-31}$$

and

$$f_1(R_2+1.5)^2 - 2\sqrt{f_1}(R_2+1.5) - .943R_2 + .586 = 0. \tag{B-32}$$

Using the quadratic equation results in

$$f_1 = 2\sqrt{f_1}(R_2+1.5) \pm \frac{\sqrt{4f_1(R_2+1.5)^2 - 4(R_2+1.5)^2(-.943R_2+.586)}}{2(R_2+1.5)^2} = 0$$

(B-33)

and simplifying gives

$$f_1 = \frac{1 \pm \sqrt{.943R_2+.414}}{(R_2+1.5)} \tag{B-34}$$

The equation for the RMR is

$$\mathrm{RMR} = \frac{\dfrac{\left(\dfrac{1}{R_2+.5}\right)^2 + \left(\dfrac{8R_2}{3(R-.5)^2(R_2+.5)^2}\right)}{\left(\dfrac{1}{.943R_2+.414} + \dfrac{8R_2}{3(R_2-.5)^2(R_2+.5)^2}\right)}}{\dfrac{1 \pm \sqrt{.943R_2+.414}}{(R_2+1.5)}} . \tag{B-35}$$

The above analysis is incomplete in that it assumes that the point $(0.889, f_1)$ is located on the point source response set. A completed analysis will include that $(0.889, f_1)$ is part of the other boundary of the complete response set. The RMR for the problem is between the two calculated RMRs from the boundary points. The optimized detector location, in addition, will have a range of calculated values.

The concept of the time variable t has been ignored (Sen-Heim 1985b). If total measurement time available is T, and the first measurement used t units of time, then T-t units are available for the next measurement. The uncertainty resulting from a second measurement for the entire time remaining, T-t, may be too large. The determination the number of detector

measurements and the partitioning of the allotted time among those measurements in order to achieve a desired level of certainty remains a relatively unexplored topic.

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BIOGRAPHICAL SKETCH

Erlen Scott enlisted in the U.S.N. in June 1976, serving aboard the U.S.S. California (CGN-36) as an engineering laboratory technician. While working as a contract technician at various nuclear power plants, he received his B.S. degree from the Regents External Degree Program, State University of New York, in May 1986. While attending classes at a branch campus Penn State, Penn State Harrisburg, and working at Limerick Generating Station, he decided to become a full-time student at the University of Florida and received an M.S. degree from that University in December 1991. His studies at the University of Florida were supported by fellowships from the Institute of Nuclear Power Operations (INPO), the Office of Civilian Radioactive Waste Management (OCRWM) and the Health Physics Society (HPS). Financial assistance was also received through the GI Bill. A Master of Engineering degree was awarded from Penn State, Harrisburg, in May 1993.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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